

Algebraic Bethe ansatz for 19-vertex models with upper triangular K-matrices

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Abstract

By means of an algebraic Bethe ansatz approach we study the Zamolodchikov-Fateev and Izergin-Korepin vertex models with non-diagonal boundaries, characterized by reflection matrices with an upper triangular form. Generalized Bethe vectors are used to diagonalize the associated transfer matrix. The eigenvalues as well as the Bethe equations are presented.

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1 Introduction

The study of integrable open vertex models in the framework of the quantum inverse scattering method was initiated in the seminal work of Sklyanin [1]. Using the so called reflection matrices [2], Sklyanin proposed a double-row transfer matrix from which integrable spin Hamiltonians with boundary couplings can be obtained. In addition to the Yang-Baxter equation, a set of compatibility conditions on the reflection matrices has to be considered in order to guarantee the commutativity of the double-row transfer matrices.

In principle, it should be possible to deal with the diagonalization problem of the double-row transfer matrix by means of the algebraic Bethe ansatz technique (ABA) [3]. It turns out, however, that the applicability of the ABA depends much on the structure of the reflection matrices. In fact, whilst the case of diagonal K -matrices is well understood, see for instance [4–8] and references therein, the general situation is much more complicated and has been a target of intense investigation over the last years. We can mention, for instance, the progress achieved in the non-diagonal case [9–11] by mapping the original problem in an equivalent one with, at best, one diagonal and one triangular boundary matrix.

A generalization of the ABA for the case where both reflection matrices have a triangular form was recently proposed [12]. We then extended this work proposing a systematic method to deal with transfer matrices possessing annihilation operators in their expressions [13]. The key point in these works is to consider a superposition of auxiliary Bethe states as eigenstates of the double-row transfer matrix. The coefficients of such linear combination can be fixed by requiring the vanishing of extra unwanted terms in the ABA analysis.

The purpose of this paper is to extend our recent work for 19-vertex models with upper triangular boundary matrices. We remember that the solution of 19-vertex models plays an essential role in the study of vertex models associated with higher rank symmetries, in both periodic and open boundary conditions [8, 14–16]. In this note, we choose as representative of this class of vertex models the Zamolodchikov-Fateev (ZF) [17] and Izergin-Korepin (IK) [18] models, for which non-diagonal reflection matrices are known [19–22].

We recall here that the study of 19-vertex models with diagonal K -matrices in the ABA framework was started in [4], subsequently considered in [23] and thereafter reviewed in [7]. These results complemented the previous one obtained by means of the analytical Bethe ansatz [24, 25], providing the corresponding Bethe vectors of the double-row transfer matrix. Very recently, the spectrum of 19-vertex models with non-diagonal K -matrices has been reported in the literature [26–28]; the construction of the respective eigenstates remains however an open problem.

This work is organized as follows. In section 2 we remind some notation and basic ingredients necessary for the boundary algebraic Bethe ansatz. We also give the R -matrix

and the upper triangular K -matrices expressions for the models we are considering. In section 3 we implement the algebraic Bethe ansatz handling the first, second and third excited states in detail. Our final remarks are given in section 4 and in the appendices necessary relations in the main text are given.

2 The models and the triangular K -matrices

The fundamental object in a two-dimensional integrable theory is the R -matrix satisfying the celebrated Yang-Baxter equation,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \quad (1)$$

where the indices ij in R_{ij} indicate the spaces where the R -matrix acts non-trivially and u, v are called spectral parameters. From the R -matrix, one can construct the so called transfer matrix,

$$t_{\text{periodic}}(u) = \text{Tr}_a[R_{a1}(u) \dots R_{aL}(u)], \quad (2)$$

where a denotes an auxiliary space and $n = 1, 2, \dots, L$ refers to a quantum vector space at the site n . In this context, the Yang-Baxter equation is a sufficient condition for the commutativity of transfer matrices in periodic boundary conditions. The integrability follows from the fact that the expansion of the transfer matrix in the spectral parameter generates an infinite number of conserved quantities.

The introduction of boundaries preserving integrability can be performed through the introduction of reflection matrices K attached in each end of the chain [1]. Indeed, the K -matrices allow us to connect the single-row monodromy matrices $T_a(u) = R_{a1}(u) \dots R_{aL}(u)$ and $\hat{T}_a(u) \equiv T_a^{-1}(-u)$ in order to define a double-row monodromy product,

$$U_a(u) = T_a(u)K^-(u)T_a^{-1}(-u) \quad (3)$$

as well as the double-row transfer matrix,

$$t(u) = \text{Tr}_a[K^+(u)T_a(u)K^-(u)\hat{T}_a(u)]. \quad (4)$$

The commutativity of the transfer matrix (4) for arbitrary spectral parameters is assured if the K -matrices satisfy the reflection equations [1, 29],

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \quad (5)$$

$$\begin{aligned} & R_{12}(v-u)K_1^{+t_1}(u)M_1^{-1}R_{21}(-u-v-2\rho)M_1K_2^{+t_2}(v) \\ &= K_2^{+t_2}(v)M_1R_{12}(-u-v-2\rho)M_1^{-1}K_1^{+t_1}(u)R_{21}(v-u) \end{aligned} \quad (6)$$

where K^+ represents the left boundary and K^- the right one. The matrix M and the parameter ρ are related to crossing-unitarity properties of the R -matrix and will be

given below. The symbol t_i denotes matrix transposition in the space i . In addition, the equations (1,5,6) imply the global relations,

$$\check{R}(u-v)T(u) \otimes T(v) = T(v) \otimes T(u)\check{R}(u-v) \quad (7)$$

and

$$R_{12}(u-v)U_1(u)R_{21}(u+v)U_2(v) = U_2(v)R_{12}(u+v)U_1(u)R_{21}(u-v), \quad (8)$$

where $\check{R}(u) = PR(u)$.

The Zamolodchikov-Fateev [17] and Izergin-Korepin [18] models are two of the known integrable trigonometric 19-vertex models (see *e.g.* [30] and references therein). The former can be considered as a direct generalization of the symmetric six-vertex model while the latter is associated with the twisted affine algebra $A_2^{(2)}$. For both models, the R -matrix has the following common structure,

$$R(u) = \left(\begin{array}{ccc|ccc|ccc} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f(u) & 0 & d(u) & 0 & h(u) & 0 & 0 \\ \hline 0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{d}(u) & 0 & e(u) & 0 & d(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0 \\ \hline 0 & 0 & \tilde{h}(u) & 0 & \tilde{d}(u) & 0 & f(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c(u) & 0 & b(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u) \end{array} \right), \quad (9)$$

where, for the ZF model we have,

$$\begin{aligned} a(u) &= 1, \quad b(u) = \frac{\sinh(u)}{\sinh(u+\eta)}, \quad c(u) = \frac{\sinh(\eta)}{\sinh(u+\eta)}, \\ \tilde{d}(u) &= d(u) = \frac{\sinh(\eta) \sinh(u)}{\sinh(u+\frac{\eta}{2}) \sinh(u+\eta)}, \quad f(u) = \frac{\sinh(u-\frac{\eta}{2}) \sinh(u)}{\sinh(u+\frac{\eta}{2}) \sinh(u+\eta)}, \\ e(u) &= \frac{\cosh(\frac{\eta}{2}-u) \cosh(u+\eta) - \cosh(\frac{\eta}{2})}{\sinh(u+\frac{\eta}{2}) \sinh(u+\eta)}, \quad \tilde{h}(u) = h(u) = \frac{\sinh(\eta) \sinh(\frac{\eta}{2})}{\sinh(u+\frac{\eta}{2}) \sinh(u+\eta)}, \end{aligned} \quad (10)$$

and, for the IK solution,

$$\begin{aligned} a(u) &= 1, \quad b(u) = \frac{\sinh(u)}{\sinh(u+\eta)}, \quad c(u) = \frac{\sinh(\eta)}{\sinh(u+\eta)}, \\ d(u) &= \frac{e^\eta \sinh(\eta) \sinh(u)}{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta)}, \quad \tilde{d}(u) = -e^{-2\eta} d(u), \\ f(u) &= \frac{\cosh(u+\frac{\eta}{2}) \sinh(u)}{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta)}, \quad e(u) = \frac{\cosh(u-\frac{\eta}{2}) \sinh(u+2\eta) - \cosh(\frac{\eta}{2}) \sinh(\eta)}{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta)}, \\ h(u) &= \frac{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta) - e^{2\eta} \cosh(u+\frac{\eta}{2}) \sinh(u)}{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta)}, \\ \tilde{h}(u) &= \frac{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta) - e^{-2\eta} \cosh(u+\frac{\eta}{2}) \sinh(u)}{\cosh(u+\frac{3\eta}{2}) \sinh(u+\eta)}. \end{aligned} \quad (11)$$

Some important properties of these R -matrices are,

1. PT symmetry: $R_{21}(u) = R_{12}^{t_1 t_2}(u)$;
2. unitarity: $R_{12}(u)R_{21}(-u) = 1 \otimes 1$; and
3. crossing-unitarity: $R_{12}^{t_1}(u)M_1 R_{12}^{t_2}(-u - 2\rho)M_1^{-1} = \zeta(u)1 \otimes 1$,

where 1 is the 3×3 identity matrix and $\zeta(u)$ is a scalar function. Considering that the matrix M satisfies $[R_{12}(u), M \otimes M] = 0$, the crossing-unitarity fix the parameter ρ for each model. We can have

$$M = 1 \quad \text{and} \quad \rho = \eta, \quad (12)$$

for the ZF model and

$$M = \begin{pmatrix} e^{-2\eta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\eta} \end{pmatrix} \quad \text{and} \quad \rho = 3\eta \quad (13)$$

for the IK model.

The reflection equations (5,6) have been analyzed in [19–22]. Here we are interested in the upper triangular solutions, which can be read from the type-I solutions classified in [22] and written as,

$$K^-(u) = \begin{pmatrix} k_{11}^-(u) & k_{12}^-(u) & k_{13}^-(u) \\ 0 & k_{22}^-(u) & k_{23}^-(u) \\ 0 & 0 & k_{33}^-(u) \end{pmatrix}, \quad (14)$$

with

$$\begin{aligned} k_{11}^-(u) &= \sinh(u + \xi_-) \sinh\left(u + \xi_- - \frac{\eta}{2}\right), & k_{22}^-(u) &= \sinh(\xi_- - u) \sinh\left(u + \xi_- - \frac{\eta}{2}\right), \\ k_{33}^-(u) &= \sinh(u - \xi_-) \sinh\left(u - \xi_- + \frac{\eta}{2}\right), & k_{12}^-(u) &= \beta_- \sinh(2u) \sinh\left(u + \xi_- - \frac{\eta}{2}\right), \\ k_{23}^-(u) &= \beta_- \sinh(2u) \sinh(\xi_- - u), & k_{13}^-(u) &= \left[\frac{\beta_-^2 \sinh(\frac{\eta}{2})}{\sinh(\eta)} \right] \sinh(2u) \sinh\left(2u - \frac{\eta}{2}\right), \end{aligned} \quad (15)$$

for the ZF model and

$$\begin{aligned} k_{11}^-(u) &= k_{33}^-(u) = \sinh\left(u + \frac{3\eta}{4} + \epsilon \frac{i\pi}{4}\right) \cosh\left(u + \frac{3\eta}{4} - \epsilon \frac{i\pi}{4}\right), \\ k_{22}^-(u) &= \sinh\left(-u + \frac{3\eta}{4} + \epsilon \frac{i\pi}{4}\right) \cosh\left(u + \frac{3\eta}{4} - \epsilon \frac{i\pi}{4}\right), \\ k_{12}^-(u) &= \beta_- \sinh(2u) \cosh\left(u + \frac{3\eta}{4} - \epsilon \frac{i\pi}{4}\right), \\ k_{23}^-(u) &= \beta_- \sinh(2u) e^{-\eta} \sinh\left(u + \frac{3\eta}{4} + \epsilon \frac{i\pi}{4}\right), \\ k_{13}^-(u) &= \left[-\frac{\beta_-^2 e^{-\eta}}{\cosh(\frac{\eta}{2})} \right] \sinh(2u) \cosh\left(u + \frac{\eta}{4} + \epsilon \frac{i\pi}{4}\right) \sinh\left(u + \frac{3\eta}{4} + \epsilon \frac{i\pi}{4}\right), \end{aligned} \quad (16)$$

for the IK model. The respective K^+ –matrices are obtained by,

$$K^+(u) = K^-(-u - \rho)M|_{\{\xi_- \rightarrow \xi_+, \beta_- \rightarrow \beta_+\}} \quad (17)$$

where M and ρ are given by (12,13).

In the above expressions, ξ_{\pm} and β_{\pm} are free constants while $\epsilon = \pm 1$. We note that the upper triangular matrices of the ZF model have two additional parameters ξ_{\pm} compared with the IK model.

3 Algebraic Bethe ansatz

In this section we apply the algebraic Bethe ansatz in order to handle the spectral problem for the transfer matrix (4). The first step to be carried out is the choice of a representation for the single-row monodromy matrices $T_a(u)$, $\hat{T}_a(u)$ and, consequently, for $U_a(u)$. In the case of 19–vertex models, a convenient one is given by the following 3×3 matrices in the auxiliary space a [4, 7, 23],

$$T_a(u) = \begin{pmatrix} T_{11}(u) & T_{12}(u) & T_{13}(u) \\ T_{21}(u) & T_{22}(u) & T_{23}(u) \\ T_{31}(u) & T_{31}(u) & T_{33}(u) \end{pmatrix}, \quad \hat{T}_a(u) = \begin{pmatrix} \hat{T}_{11}(u) & \hat{T}_{12}(u) & \hat{T}_{13}(u) \\ \hat{T}_{21}(u) & \hat{T}_{22}(u) & \hat{T}_{23}(u) \\ \hat{T}_{31}(u) & \hat{T}_{31}(u) & \hat{T}_{33}(u) \end{pmatrix}, \quad (18)$$

with operator entries defined on the Hilbert space $\otimes_{i=1}^L \mathbb{C}^3$. The double-row monodromy matrix can be thus written as,

$$U_a(u) = \begin{pmatrix} \mathcal{A}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{A}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{A}_3(u) \end{pmatrix}, \quad (19)$$

where the operators \mathcal{A}_j , \mathcal{B}_j and \mathcal{C}_j are given in terms of T_{ij} and \hat{T}_{ij} . This step is important since it provides commutation relations between the entries of (19) thanks to the relation (8).

We next have to consider a reference state as well as the corresponding action of the U –operators on it. Due to the structure of the right K –matrices (14), the pseudovacuum defined by

$$\Psi_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{(1)} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{(2)} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{(L)}, \quad (20)$$

turns out to be a good choice of reference state [11]. In fact, taking into account the exchange relations between the matrix elements of $T(u)$ and $\hat{T}(u)$, provided by equation (7), as well as the expressions for $T_{ij}(u)\Psi_0$ and $\hat{T}_{ij}(u)\Psi_0$, we are able to calculate¹,

$$\mathcal{D}_j(u)\Psi_0 = \Delta_j(u)\Psi_0 \quad \text{for } j = 1, 2, 3 \quad (21)$$

¹For more details on this calculation see for instance [7].

and

$$\mathcal{C}_j(u)\Psi_0 = 0 \quad \text{for } j = 1, 2, 3, \quad (22)$$

where

$$\begin{aligned} \Delta_1(u) &= k_{11}^-(u), \quad \Delta_2(u) = [k_{22}^-(u) - f_1(u)k_{11}^-(u)] b(u)^{2L}, \\ \Delta_3(u) &= [k_{33}^-(u) - f_4(u)k_{11}^-(u) - f_3(u)k_{22}^-(u)] f(u)^{2L}. \end{aligned} \quad (23)$$

The shifted operators $\mathcal{D}_j(u)$ are conveniently defined by,

$$\mathcal{D}_1(u) = \mathcal{A}_1(u), \quad \mathcal{D}_2(u) = \mathcal{A}_2(u) - f_1(u)\mathcal{A}_1(u), \quad \mathcal{D}_3(u) = \mathcal{A}_3(u) - f_2(u)\mathcal{A}_1 - f_3(u)\mathcal{D}_2(u), \quad (24)$$

where the auxiliary functions f are given by,

$$\begin{aligned} f_1(u) &= c(2u), \quad f_2(u) = \tilde{h}(2u), \quad f_3(u) = \frac{c(2u) [\tilde{h}(2u) - 1]}{c(2u)^2 - e(2u)}, \\ f_4(u) &= \frac{c(2u)^2 - \tilde{h}(2u)e(2u)}{c(2u)^2 - e(2u)}. \end{aligned} \quad (25)$$

In terms of the operators (24), the transfer matrix expression can be decomposed in two parts as follows,

$$t(u) = t_d(u) + t_u(u) \quad (26)$$

where

$$t_d(u) = \omega_1(u)\mathcal{D}_1(u) + \omega_2(u)\mathcal{D}_2(u) + \omega_3(u)\mathcal{D}_3(u) \quad (27)$$

and,

$$t_u(u) = k_{12}^+(u)\mathcal{C}_1(u) + k_{13}^+(u)\mathcal{C}_2(u) + k_{23}^+(u)\mathcal{C}_3(u). \quad (28)$$

with

$$\begin{aligned} \omega_1(u) &= k_{11}^+(u) + f_1(u)k_{22}^+(u) + f_2(u)k_{33}^+(u), \quad \omega_2(u) = k_{22}^+(u) + f_3(u)k_{33}^+(u), \\ \omega_3(u) &= k_{33}^+(u). \end{aligned} \quad (29)$$

Since the operators $\mathcal{C}_j(u)$ destroy the state (20) we have $t_u(u)\Psi_0 = 0$. Thus, the state (20) is a common eigenstate of the transfer matrices (26) and (27) with eigenvalue,

$$\Lambda_0(u) = \omega_1(u)\Delta_1(u) + \omega_2(u)\Delta_2(u) + \omega_3(u)\Delta_3(u). \quad (30)$$

Within the ABA framework, the excited states of the term $t_d(u)$ can be generated by applying creation operators on the reference state (20) [4, 7, 23]. More precisely, the eigenstates of $t_d(u)$ are constructed by applying both $\mathcal{B}_1(u)$ and $\mathcal{B}_2(u)$ operators on Ψ_0 . For instance, the one-particle $\Psi_1(u_1)$ and the two-particle $\Psi_2(u_1, u_2)$ states are given by,

$$\Psi_1(u_1) = \mathcal{B}_1(u_1)\Psi_0, \quad (31)$$

$$\Psi_2(u_1, u_2) = \mathcal{B}_1(u_1)\mathcal{B}_1(u_2)\Psi_0 - \Gamma_2^{(2)}(u_1, u_2)\mathcal{B}_2(u_1)\Psi_0 \quad (32)$$

where the function $\Gamma_2^{(2)}(u_1, u_2)$ is fixed by requiring the symmetry

$$\Psi_2(u_2, u_1) = \Omega(u_1, u_2)\Psi_2(u_1, u_2), \quad \Omega(u_1, u_2) = 1/e_{01}(u_1, u_2) \quad (33)$$

as well as the commutation relation (A.7). Moreover, the multi-particle states satisfy a Tarasov-like recurrence relation [31], namely,

$$\begin{aligned} \Psi_n(u_1, \dots, u_n) &= \mathcal{B}_1(u_1)\Psi_{n-1}(u_2, \dots, u_n) \\ &- \mathcal{B}_2(u_1) \sum_{i=2}^n \Gamma_i^{(n)}(u_1, \dots, u_n) \Psi_{n-2}(u_2, \dots, \hat{u}_i, \dots, u_n) \end{aligned} \quad (34)$$

with the function Γ_i given by,

$$\begin{aligned} \Gamma_i^{(n)}(u_1, \dots, u_n) &= \prod_{j=2, j < i}^n \Omega(u_i, u_j) \\ &\times \left\{ \Delta_1(u_i) e_{04}(u_1, u_i) \prod_{k=2, k \neq i}^n a_{11}(u_i, u_k) + \Delta_2(u_i) e_{05}(u_1, u_i) \prod_{k=2, k \neq i}^n a_{21}(u_i, u_k) \right\}. \end{aligned} \quad (35)$$

where, as usual, the notation \hat{u}_i means that the rapidity u_i is absent in the function.

In our case, however, the transfer matrix possesses the annihilation operators $\mathcal{C}_{1,2,3}(u)$ in its expression and, as a result, we have to seek for more intricate eigenstates. We propose that a superposition of the states (34) is needed to diagonalize the full transfer matrix (26), namely,

$$\Phi_n = \sum_{k=0}^n g^{(k)} \Psi_k. \quad (36)$$

Therefore, the main task in this work is to fix the formulas for the g -coefficients in (36). To this end, we shall need to calculate the action of both $t_d(u)$ and $t_u(u)$ on the vectors Ψ_n . Since the action of the shifted operators as well as of the annihilation operators on the reference are known (21,22), we will need to move the operators \mathcal{D}_j and \mathcal{C}_j over the creation operators \mathcal{B}_j . This can be achieved by the repeated use of the commutation relations given in appendix A which are derived from equation (8). After a very long though straightforward calculation, up to $n = 3$, we are able to propose that the action $t_d(u)\Psi_n$ can be written as,

$$\begin{aligned} t_d(u)\Psi_n(u_1, \dots, u_n) &= \Lambda_n(u, u_1, \dots, u_n)\Psi_n(u_1, \dots, u_n) \\ &+ \mathcal{B}_1(u) \sum_{j=1}^n \mathcal{F}_j^{(n)}(u, u_1, \dots, u_n) \Psi_{n-1}(u_1, \dots, \hat{u}_j, \dots, u_n) \\ &+ \mathcal{B}_3(u) \sum_{j=1}^n \mathcal{G}_j^{(n)}(u, u_1, \dots, u_n) \Psi_{n-1}(u_1, \dots, \hat{u}_j, \dots, u_n) \\ &+ \mathcal{B}_2(u) \sum_{j < k}^n \mathcal{H}_{jk}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-2}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, u_n), \end{aligned} \quad (37)$$

while $t_u(u)\Psi_n$ has a more involved structure and it is given by,

$$\begin{aligned}
t_u(u)\Psi_n(u_1, \dots, u_n) &= \sum_{j=1}^n \mathcal{T}_j^{(n)}(u, u_1, \dots, u_n) \Psi_{n-1}(u_1, \dots, \hat{u}_j, \dots, u_n) \\
&+ \sum_{j < k}^n \mathcal{U}_{jk}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-2}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, u_n) \\
&+ \mathcal{B}_1(u) \sum_{j < k}^n \mathcal{V}_{jk}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-2}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, u_n) \\
&+ \mathcal{B}_3(u) \sum_{j < k}^n \mathcal{W}_{jk}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-2}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, u_n) \\
&+ \mathcal{B}_1(u) \sum_{j < k < \ell}^n \mathcal{X}_{jk\ell}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-3}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, \hat{u}_\ell, \dots, u_n) \\
&+ \mathcal{B}_3(u) \sum_{j < k < \ell}^n \mathcal{Y}_{jk\ell}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-3}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, \hat{u}_\ell, \dots, u_n) \\
&+ \mathcal{B}_2(u) \sum_{j < k < \ell}^n \mathcal{Z}_{jk\ell}^{(n)}(u, u_1, \dots, u_n) \Psi_{n-3}(u_1, \dots, \hat{u}_j, \dots, \hat{u}_k, \dots, \hat{u}_\ell, \dots, u_n). \quad (38)
\end{aligned}$$

where the functions $\mathcal{F}, \mathcal{G}, \dots, \mathcal{Z}$ entering equations (37) and (38) are given in appendix B. We now use the above evaluations in order to consider in detail the first, second and third excited states, which allow us to propose a general formula for the g -coefficients.

3.1 First excited state

Considering our previous discussion we suppose as the first excited state of (26) the linear combination,

$$\Phi_1(u_1) = \Psi_1(u_1) + g(u_1)\Psi_0. \quad (39)$$

with $g(u_1)$ to be fixed *a posteriori*. Acting with $t(u)$ on it we obtain,

$$t(u)\Phi_1(u_1) = t_d(u)\Psi_1(u_1) + g(u_1)t_d(u)\Psi_0 + t_u(u)\Psi_1(u_1), \quad (40)$$

and, using $n = 1$ in the formulas (37,38), we have,

$$\begin{aligned}
t(u)\Phi_1(u_1) &= \Lambda_1(u, u_1)\Phi_1(u_1) + \mathcal{F}_1^{(1)}(u, u_1)\mathcal{B}_1(u)\Psi_0 + \mathcal{G}_1^{(1)}(u, u_1)\mathcal{B}_3(u)\Psi_0 \\
&+ \left\{ g(u_1) [\Lambda_0(u) - \Lambda_1(u, u_1)] + \mathcal{T}_1^{(1)}(u, u_1) \right\} \Psi_0
\end{aligned} \quad (41)$$

where

$$\Lambda_1(u, u_1) = \omega_1(u)\Delta_1(u)a_{11}(u, u_1) + \omega_2(u)\Delta_2(u)(u)a_{21}(u, u_1) + \omega_3(u)\Delta_3(u)a_{31}(u, u_1). \quad (42)$$

We see that $\Phi_1(u_1)$ will be an eigenstate of $t(u)$ with eigenvalue $\Lambda_1(u, u_1)$ provided that the coefficients of the unwanted states $\mathcal{B}_1(u)\Psi_0$, $\mathcal{B}_3(u)\Psi_0$ and Ψ_0 vanish. The nullity requirement of $\mathcal{F}_1^{(1)}(u, u_1)$ lead us to the constraint,

$$\frac{\Delta_1(u_1)}{\Delta_2(u_1)} = -\frac{Q_2^{\mathcal{F}}(u, u_1)}{Q_1^{\mathcal{F}}(u, u_1)} \quad (43)$$

and, due to the identity,

$$\frac{Q_2^{\mathcal{F}}(u, u_1)}{Q_1^{\mathcal{F}}(u, u_1)} = \frac{Q_2^{\mathcal{G}}(u, u_1)}{Q_1^{\mathcal{G}}(u, u_1)} \quad (44)$$

the function $\mathcal{G}_1^{(1)}(u, u_1)$ also vanishes if (43) is satisfied. Considering the explicit expressions of the weights we note that the dependence on the parameter u disappears in the right-hand side of (43) and, as a result, we can write the Bethe equation constraining the rapidity u_1 as,

$$\frac{\Delta_1(u_1)}{\Delta_2(u_1)} = -\Theta(u_1) \quad (45)$$

where, for the ZF model,

$$\Theta(u_1) = \frac{\sinh(2u_1 + \eta) \sinh(u_1 + \xi_+ + \frac{\eta}{2})}{\sinh(2u_1) \sinh(u_1 + \frac{\eta}{2} - \xi_+)} \quad (46)$$

and, for the IK model,

$$\Theta(u_1) = -\frac{\sinh(2u_1 + \eta) \sinh(u_1 + \frac{\eta}{4} - \epsilon \frac{3i\pi}{4})}{\sinh(2u_1) \sinh(u_1 + \frac{3\eta}{4} - \epsilon \frac{i\pi}{4})} \quad (47)$$

We are left then with the coefficient of Ψ_0 in equation (41). It can be used to linearly extract an expression for $g(u_1)$,

$$g(u_1) = \frac{\mathcal{T}_1^{(1)}(u, u_1)}{\Lambda_1(u, u_1) - \Lambda_0(u)}. \quad (48)$$

To be an eigenstate of $t(u)$ the expression (39) cannot depend on the spectral parameter u and thus (48) seems inconsistent. However, the dependence on u disappears once we take into account the Bethe equation (45). Indeed, the cumbersome expression (48), if the rapidity u_1 is a solution of (45), is simplified to

$$g(u_1) = \beta_+ \left[\frac{\sinh(2u_1 + \eta)}{\sinh(u_1 + \frac{\eta}{2} - \xi_+)} \right] \Delta_2(u_1), \quad (49)$$

in the case of the ZF model and

$$g(u_1) = \beta_+ \left[\frac{\sinh(2u_1 + \eta)}{\sinh(u_1 + \frac{3\eta}{4} - \epsilon \frac{i\pi}{4})} \right] \Delta_2(u_1), \quad (50)$$

for the IK model.

We have thus determined the first excited state. We observe that the transfer matrix $t(u)$ shares with $t_d(u)$ the eigenvalue expression (42) as well as the Bethe equation (45).

3.2 Second excited state

For the second excited state the ansatz is,

$$\begin{aligned}\Phi_2(u_1, u_2) &= \Psi_2(u_1, u_2) \\ &+ g_2^{(1)}(u_1, u_2)\Psi_1(u_1) + g_1^{(1)}(u_1, u_2)\Psi_1(u_2) \\ &+ g_{12}^{(0)}(u_1, u_2)\Psi_0\end{aligned}\tag{51}$$

with the coefficients $g_{1,2}^{(1)}(u_1, u_2)$ and $g_{12}^{(0)}(u_1, u_2)$ to be fixed in what follows.

As before, we need to know the action of $t(u)$ on the state (51). This can be done by setting $n = 2$ in the expressions (37,38). As a result we get the following off-shell expression,

$$\begin{aligned}t(u)\Phi_2(u_1, u_2) &= \Lambda_2(u, u_1, u_2)\Phi_2(u_1, u_2) \\ &+ \mathcal{F}_1^{(2)}(u, u_1, u_2)\mathcal{B}_1(u)\Psi_1(u_2) + \mathcal{F}_2^{(2)}(u, u_1, u_2)\mathcal{B}_1(u)\Psi_1(u_1) \\ &+ \mathcal{G}_1^{(2)}(u, u_1, u_2)\mathcal{B}_3(u)\Psi_1(u_2) + \mathcal{G}_2^{(2)}(u, u_1, u_2)\mathcal{B}_3(u)\Psi_1(u_1) \\ &+ \mathcal{H}_{12}^{(2)}(u, u_1, u_2)\mathcal{B}_2(u)\Psi_0 \\ &+ \left\{ g_2^{(1)}(u_1, u_2)\mathcal{F}_1^{(1)}(u, u_1) + g_1^{(1)}(u_1, u_2)\mathcal{F}_1^{(1)}(u, u_2) + \mathcal{V}_{12}^{(2)}(u, u_1, u_2) \right\} \mathcal{B}_1(u)\Psi_0 \\ &+ \left\{ g_2^{(1)}(u_1, u_2)\mathcal{G}_1^{(1)}(u, u_1) + g_1^{(1)}(u_1, u_2)\mathcal{G}_1^{(1)}(u, u_2) + \mathcal{W}_{12}^{(2)}(u, u_1, u_2) \right\} \mathcal{B}_3(u)\Psi_0 \\ &+ \left\{ g_1^{(1)}(u_1, u_2) [\Lambda_1(u, u_2) - \Lambda_2(u, u_1, u_2)] + \mathcal{T}_1^{(2)}(u, u_1, u_2) \right\} \Psi_1(u_2) \\ &+ \left\{ g_2^{(1)}(u_1, u_2) [\Lambda_1(u, u_1) - \Lambda_2(u, u_1, u_2)] + \mathcal{T}_2^{(2)}(u, u_1, u_2) \right\} \Psi_1(u_1) \\ &+ \left\{ g_{12}^{(0)}(u_1, u_2) [\Lambda_0(u) - \Lambda_2(u, u_1, u_2)] \right. \\ &\left. + g_1^{(1)}(u_1, u_2)\mathcal{T}_1^{(1)}(u, u_2) + g_2^{(1)}(u_1, u_2)\mathcal{T}_1^{(1)}(u, u_1) + \mathcal{U}_{12}^{(2)}(u, u_1, u_2) \right\} \Psi_0\end{aligned}\tag{52}$$

We observe that the first five unwanted terms in (52) coincide with the unwanted terms of the action of the diagonal transfer matrix $t_d(u)$ on the state Ψ_2 (32). Therefore, their vanishing lead to the Bethe equations for the rapidities u_1 and u_2 , namely

$$\frac{\Delta_1(u_j)}{\Delta_2(u_j)} = -\Theta(u_j) \prod_{k=1, k \neq j}^2 \frac{a_{21}(u_j, u_k)}{a_{11}(u_j, u_k)} \quad \text{for } j = 1, 2.\tag{53}$$

We remark that the constraints (53) are obtained from the equations $\mathcal{F}_j^{(2)}(u, u_1, u_2) = 0$ taking into account the explicit formulas for the Boltzmann weights. Due to identities provenient from the Yang-Baxter and reflection algebras, we note that the coefficients $\mathcal{G}_j^{(2)}(u, u_1, u_2)$ and $\mathcal{H}_{12}^{(2)}(u, u_1, u_2)$ also vanish if (53) are valid.

The other coefficients are used in order to extract the expressions for the unknown functions in the ansatz state (51). The vanishing requirement of the coefficients of $\Psi_1(u_1)$, $\Psi_1(u_2)$ and Ψ_0 allows us to write,

$$g_1^{(1)}(u_1, u_2) = \frac{\mathcal{T}_1^{(2)}(u, u_1, u_2)}{\Lambda_2(u, u_1, u_2) - \Lambda_1(u, u_2)}, \quad g_2^{(1)}(u_1, u_2) = \frac{\mathcal{T}_2^{(2)}(u, u_1, u_2)}{\Lambda_2(u, u_1, u_2) - \Lambda_1(u, u_1)}\tag{54}$$

and

$$g_{12}^{(0)}(u_1, u_2) = \frac{g_1^{(1)}(u_1, u_2)\mathcal{T}_1^{(1)}(u, u_2) + g_2^{(1)}(u_1, u_2)\mathcal{T}_1^{(1)}(u, u_1) + \mathcal{U}_{12}^{(2)}(u, u_1, u_2)}{\Lambda_2(u, u_1, u_2) - \Lambda_0(u)}. \quad (55)$$

Similarly to the first excited state, the resulting expressions for coefficients (54,55) contain the spectral parameter u . Once again, this situation can be overcome if we consider the expression for the ratio Δ_1/Δ_2 constrained by the Bethe equations (53). After some cumbersome manipulation we find that the expressions (54,55) simplify to the following factorized structure,

$$\begin{aligned} g_1^{(1)}(u_1, u_2) &= g(u_1)a_{21}(u_1, u_2), \\ g_2^{(1)}(u_1, u_2) &= g(u_2)a_{21}(u_2, u_1)\Omega(u_2, u_1), \\ g_{12}^{(0)}(u_1, u_2) &= g(u_1)g(u_2)s(u_1, u_2) \end{aligned} \quad (56)$$

where $g(u_i)$ are the functions (49) or (50) provenient from the first excited state analysis and,

$$s(u_1, u_2) = \frac{\sinh(u_1 + u_2) \sinh(u_1 - u_2 - \eta) \sinh(u_1 + u_2 + \frac{3\eta}{2})}{\sinh(u_1 - u_2 - \frac{\eta}{2}) \sinh(u_1 + u_2 + \frac{\eta}{2})^2} \quad (57)$$

for the ZF model and

$$\begin{aligned} s(u_1, u_2) &= \frac{\cosh(u_1 + u_2 + \frac{\eta}{2}) \sinh(u_1 + u_2 + 2\eta)}{\sinh(u_1 + u_2) \cosh(u_1 - u_2 + \frac{\eta}{2}) \cosh(u_1 + u_2 + \frac{3\eta}{2})^2} \\ &\times \left[\cosh\left(u_1 + \frac{\eta}{4} - i\epsilon\frac{\pi}{4}\right) \cosh\left(u_1 + \frac{3\eta}{4} + i\epsilon\frac{\pi}{4}\right) \right. \\ &\quad \left. + \cosh\left(u_2 - \frac{\eta}{4} + i\epsilon\frac{\pi}{4}\right) \cosh\left(u_2 + \frac{5\eta}{4} - i\epsilon\frac{\pi}{4}\right) \right] \end{aligned} \quad (58)$$

for the IK solution. We observe the appearing of the factor $\Omega(u_1, u_2)$ in the coefficient $g_2^{(1)}(u_1, u_2)$. As a consequence, we also have the symmetry property (33) for the generalized state $\Phi_2(u_1, u_2)$.

At this point, we use the expressions (56) in the remaining unwanted terms of (52) and, taking into account the constraints on u_1 and u_2 dictated by the Bethe equations (60), we can check by direct computation that all of them are automatically cancelled. In this way, we conclude that the state (51) with coefficients (56) is an eigenstate of (26) with eigenvalue $\Lambda_2(u, u_1, u_2)$ (B.1).

3.3 Third excited state

We proceed in a similar way for the third excited state. The ansatz now is given by,

$$\begin{aligned} \Phi_3(u_1, u_2, u_3) &= \Psi_3(u_1, u_2, u_3) + g_3^{(2)}(u_1, u_2, u_3)\Psi_2(u_1, u_2) \\ &+ g_2^{(2)}(u_1, u_2, u_3)\Psi_2(u_1, u_3) + g_1^{(2)}(u_1, u_2, u_3)\Psi_2(u_2, u_3) \\ &+ g_{23}^{(1)}(u_1, u_2, u_3)\Psi_1(u_1) + g_{13}^{(1)}(u_1, u_2, u_3)\Psi_1(u_2) + g_{12}^{(1)}(u_1, u_2, u_3)\Psi_1(u_3) \\ &+ g_{123}^{(0)}(u_1, u_2, u_3)\Psi_0 \end{aligned} \quad (59)$$

where the coefficients $g^{(k)}(u_1, u_2, u_3)$ will be determined in the forthcoming analysis.

Once again, we need to evaluate the action of $t(u)$ on the ansatz state (59). For this end we use $n = 3$ in the expansions (37,38) and, as a result, we note the appearing of many cumbersome unwanted terms. For this reason, we omit their explicit formulas here. It turns out that the Bethe equations can be obtained from the coefficients of the unwanted states $\mathcal{B}_1(u)\Psi_2(u_j, u_k)$, and it read

$$\frac{\Delta_1(u_j)}{\Delta_2(u_j)} = -\Theta(u_j) \prod_{k=1, k \neq j}^3 \frac{a_{21}(u_j, u_k)}{a_{11}(u_j, u_k)} \quad \text{for } j = 1, 2, 3. \quad (60)$$

We then choose the simplest unwanted terms to obtain the g -functions. For the amplitudes of order two, namely, $g^{(2)}(u_1, u_2, u_3)$, we use the coefficients of $\Psi_2(u_j, u_k)$ and, after taking into account the Bethe equations (60), we find,

$$\begin{aligned} g_3^{(2)}(u_1, u_2, u_3) &= g(u_3)a_{21}(u_3, u_1)a_{21}(u_3, u_2)\Omega(u_3, u_1)\Omega(u_3, u_2), \\ g_2^{(2)}(u_1, u_2, u_3) &= g(u_2)a_{21}(u_2, u_1)a_{21}(u_2, u_3)\Omega(u_2, u_1), \\ g_1^{(2)}(u_1, u_2, u_3) &= g(u_1)a_{21}(u_1, u_2)a_{21}(u_1, u_3). \end{aligned} \quad (61)$$

The functions $g^{(1)}(u_1, u_2, u_3)$ are derived from the coefficients of $\Psi_1(u_j)$ while $g^{(0)}(u_1, u_2, u_3)$ is obtained from the coefficient of Ψ_0 . Using the Bethe equations (60), they acquire the following structure,

$$\begin{aligned} g_{23}^{(1)}(u_1, u_2, u_3) &= g(u_2)g(u_3)a_{21}(u_2, u_1)a_{21}(u_3, u_1)s(u_2, u_3)\Omega(u_2, u_1)\Omega(u_3, u_1), \\ g_{13}^{(1)}(u_1, u_2, u_3) &= g(u_1)g(u_3)a_{21}(u_1, u_2)a_{21}(u_3, u_2)s(u_1, u_3)\Omega(u_3, u_2), \\ g_{12}^{(1)}(u_1, u_2, u_3) &= g(u_1)g(u_2)a_{21}(u_2, u_3)a_{21}(u_1, u_3)s(u_1, u_2), \\ g_{123}^{(0)}(u_1, u_2, u_3) &= g(u_1)g(u_2)g(u_3)s(u_1, u_2)s(u_1, u_3)s(u_2, u_3). \end{aligned} \quad (62)$$

Finally, we can see by direct computation that all the other unwanted terms are cancelled provided that we take into account the expressions (61,62) in addition to the Bethe equations (60). Therefore, the vector (59) is an eigenstate of the transfer matrix (26) with energy $\Lambda_3(u, u_1, u_2, u_3)$ (B.1).

3.4 General excited state

Considering the results of the previous subsections we are able to propose a solution of the spectral problem associated with the transfer matrix (26): the n th excited eigenstate is given by

$$\begin{aligned} \Phi_n(u_1, \dots, u_n) &= \Psi_n(u_1, \dots, u_n) \\ + \sum_{k=0}^{n-1} \sum_{\ell_1 < \dots < \ell_{n-k}=1}^n g_{\ell_1, \dots, \ell_{n-k}}^{(k)}(u_1, \dots, u_n) &\Psi_k(u_1, \dots, \hat{u}_{\ell_1}, \dots, \hat{u}_{\ell_{n-k}}, \dots, u_n), \end{aligned} \quad (63)$$

where the Ψ_n vectors are obtained by the recurrence relation (34) and the g -functions have the following expression,

$$g_{\ell_1, \dots, \ell_{n-k}}^{(k)}(u_1, \dots, u_n) = \prod_{m \in \bar{\ell}} g(u_m) \prod_{m' \in \bar{\ell}, m' < m} s(u_{m'}, u_m) \prod_{m''=1, m'' \notin \bar{\ell}}^n a_{21}(u_m, u_{m''}) \tilde{\Omega}_{m, m''} \quad (64)$$

with $\bar{\ell} = \{\ell_1, \dots, \ell_{n-k}\}$ and

$$\tilde{\Omega}_{m, m''} = \begin{cases} \Omega(u_m, u_{m''}), & \text{if } m > m'' \\ 1, & \text{otherwise.} \end{cases} \quad (65)$$

The associated eigenvalue is given by,

$$\Lambda_n(u, u_1, \dots, u_n) = \sum_{\alpha=1}^3 \omega_{\alpha}(u) \Delta_{\alpha}(u) \prod_{j=1}^n a_{\alpha 1}(u, u_j), \quad (66)$$

while the Bethe rapidities have to satisfy,

$$\frac{\Delta_1(u_j)}{\Delta_2(u_j)} = -\Theta(u_j) \prod_{k=1, k \neq j}^n \frac{a_{21}(u_j, u_k)}{a_{11}(u_j, u_k)}, \quad (67)$$

for $j = 1, \dots, n$.

4 Conclusion

By means of a generalized Bethe ansatz approach, we have presented a solution of the open Zamolodchikov-Fateev and Izergin-Korepin vertex models with triangular boundaries. Remarkably, the eigenvalues of the corresponding transfer matrix (66) as well as the Bethe equations (67) are exactly the same of the diagonal boundary case. On the other hand, the eigenstates have to be constructed from a linear superposition (63) of auxiliary Bethe states (34). These results show that such structure is not restricted to the six-vertex model [12, 13] and could be extended for open vertex models associated with higher rank algebras.

Our results may also be useful in the study of 19-vertex models with complete boundary matrices in the framework of the ABA. More specifically, it would be interesting to generalize the approach of the recent work [32] at least for rational versions of the models considered here.

Another interesting further direction of investigation is a thorough analysis of the off-shell structure dictated by equations (37,38). In fact, the quasi-classical limit of the ABA solution, along the lines of [33], can possibly lead to generalized three-state Gaudin magnets and Knizhnik-Zamolodchikov equations.

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A Commutation relations

In this section we present exchange relations coming from equation (8) which are necessary to evaluate (37) and (38). We have,

$$\begin{aligned}
\mathcal{D}_1(u)\mathcal{B}_1(v) &= a_{11}(u, v)\mathcal{B}_1(v)\mathcal{D}_1(u) + a_{12}(u, v)\mathcal{B}_1(u)\mathcal{D}_1(v) + a_{13}(u, v)\mathcal{B}_1(u)\mathcal{D}_2(v) \\
&+ a_{14}(u, v)\mathcal{B}_2(u)\mathcal{C}_1(v) + a_{15}(u, v)\mathcal{B}_2(u)\mathcal{C}_3(v) + a_{16}(u, v)\mathcal{B}_2(v)\mathcal{C}_1(u),
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\mathcal{D}_2(u)\mathcal{B}_1(v) &= a_{21}(u, v)\mathcal{B}_1(v)\mathcal{D}_2(u) + a_{22}(u, v)\mathcal{B}_1(u)\mathcal{D}_1(v) + a_{23}(u, v)\mathcal{B}_1(u)\mathcal{D}_2(v) \\
&+ a_{24}(u, v)\mathcal{B}_3(u)\mathcal{D}_1(v) + a_{25}(u, v)\mathcal{B}_3(u)\mathcal{D}_2(v) + a_{26}(u, v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
&+ a_{27}(u, v)\mathcal{B}_2(u)\mathcal{C}_3(v) + a_{28}(u, v)\mathcal{B}_2(v)\mathcal{C}_1(u) + a_{29}(u, v)\mathcal{B}_2(v)\mathcal{C}_3(u) \\
&+ a_{210}(u, v)\mathcal{B}_1(v)\mathcal{D}_1(u),
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\mathcal{D}_3(u)\mathcal{B}_1(v) &= a_{31}(u, v)\mathcal{B}_1(v)\mathcal{D}_3(u) + a_{32}(u, v)\mathcal{B}_1(u)\mathcal{D}_1(v) + a_{33}(u, v)\mathcal{B}_1(u)\mathcal{D}_2(v) \\
&+ a_{34}(u, v)\mathcal{B}_3(u)\mathcal{D}_1(v) + a_{35}(u, v)\mathcal{B}_3(u)\mathcal{D}_2(v) + a_{36}(u, v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
&+ a_{37}(u, v)\mathcal{B}_2(u)\mathcal{C}_3(v) + a_{38}(u, v)\mathcal{B}_2(v)\mathcal{C}_1(u) + a_{39}(u, v)\mathcal{B}_2(v)\mathcal{C}_3(u) \\
&+ a_{310}(u, v)\mathcal{B}_1(v)\mathcal{D}_1(u) + a_{311}(u, v)\mathcal{B}_1(v)\mathcal{D}_2(u),
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\mathcal{C}_1(u)\mathcal{B}_1(v) &= c_{11}(u, v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{12}(u, v)\mathcal{B}_1(v)\mathcal{C}_3(u) + c_{13}(u, v)\mathcal{B}_1(u)\mathcal{C}_3(v) \\
&+ c_{14}(u, v)\mathcal{B}_3(u)\mathcal{C}_3(v) + c_{15}(u, v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{16}(u, v)\mathcal{D}_1(v)\mathcal{D}_1(u) \\
&+ c_{17}(u, v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{18}(u, v)\mathcal{D}_1(u)\mathcal{D}_1(v) + c_{19}(u, v)\mathcal{D}_1(u)\mathcal{D}_2(v) \\
&+ c_{110}(u, v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{111}(u, v)\mathcal{D}_2(u)\mathcal{D}_2(v),
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\mathcal{C}_2(u)\mathcal{B}_1(v) &= y_{11}(u,v)\mathcal{B}_1(v)\mathcal{C}_2(u) + y_{12}(u,v)\mathcal{B}_3(v)\mathcal{C}_2(u) + y_{13}(u,v)\mathcal{B}_1(u)\mathcal{C}_2(v) \\
&+ y_{14}(u,v)\mathcal{B}_3(u)\mathcal{C}_2(v) + y_{15}(u,v)\mathcal{D}_1(v)\mathcal{C}_1(u) + y_{16}(u,v)\mathcal{D}_2(v)\mathcal{C}_1(u) \\
&+ y_{17}(u,v)\mathcal{D}_1(v)\mathcal{C}_3(u) + y_{18}(u,v)\mathcal{D}_2(v)\mathcal{C}_3(u) + y_{19}(u,v)\mathcal{D}_1(u)\mathcal{C}_1(v) \\
&+ y_{110}(u,v)\mathcal{D}_2(u)\mathcal{C}_1(v) + y_{111}(u,v)\mathcal{D}_3(u)\mathcal{C}_1(v) + y_{112}(u,v)\mathcal{D}_1(u)\mathcal{C}_3(v) \\
&+ y_{113}(u,v)\mathcal{D}_2(u)\mathcal{C}_3(v) + y_{114}(u,v)\mathcal{D}_3(u)\mathcal{C}_3(v), \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_3(u)\mathcal{B}_1(v) &= c_{21}(u,v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{22}(u,v)\mathcal{B}_1(v)\mathcal{C}_3(u) + c_{23}(u,v)\mathcal{B}_1(u)\mathcal{C}_3(v) \\
&+ c_{24}(u,v)\mathcal{B}_3(u)\mathcal{C}_3(v) + c_{25}(u,v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{26}(u,v)\mathcal{D}_1(v)\mathcal{D}_1(u) \\
&+ c_{27}(u,v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{28}(u,v)\mathcal{D}_1(v)\mathcal{D}_3(u) + c_{29}(u,v)\mathcal{D}_1(u)\mathcal{D}_1(v) \\
&+ c_{210}(u,v)\mathcal{D}_1(u)\mathcal{D}_2(v) + c_{211}(u,v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{212}(u,v)\mathcal{D}_2(u)\mathcal{D}_2(v) \\
&+ c_{213}(u,v)\mathcal{D}_3(u)\mathcal{D}_1(v) + c_{214}(u,v)\mathcal{D}_3(u)\mathcal{D}_2(v), \tag{A.6}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_1(u)\mathcal{B}_1(v) &= e_{01}(u,v)\mathcal{B}_1(v)\mathcal{B}_1(u) + e_{02}(u,v)\mathcal{B}_2(v)\mathcal{D}_2(u) + e_{03}(u,v)\mathcal{B}_2(v)\mathcal{D}_1(u) \\
&+ e_{04}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + e_{05}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v), \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_1(u)\mathcal{B}_3(v) &= d_{11}(u,v)\mathcal{B}_3(v)\mathcal{B}_1(u) + d_{12}(u,v)\mathcal{B}_1(v)\mathcal{B}_1(u) + d_{13}(u,v)\mathcal{B}_2(v)\mathcal{D}_1(u) \\
&+ d_{14}(u,v)\mathcal{B}_2(v)\mathcal{D}_2(u) + d_{15}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + d_{16}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&+ d_{17}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v), \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_1(u)\mathcal{B}_2(v) &= b_{11}(u,v)\mathcal{B}_2(v)\mathcal{D}_1(u) + b_{12}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{13}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&+ b_{14}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{15}(u,v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{16}(u,v)\mathcal{B}_1(u)\mathcal{B}_3(v), \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_2(u)\mathcal{B}_2(v) &= b_{21}(u,v)\mathcal{B}_2(v)\mathcal{D}_2(u) + b_{22}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{23}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&+ b_{24}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{25}(u,v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{26}(u,v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\
&+ b_{27}(u,v)\mathcal{B}_3(u)\mathcal{B}_1(v) + b_{28}(u,v)\mathcal{B}_3(u)\mathcal{B}_3(v) + b_{29}(u,v)\mathcal{B}_2(v)\mathcal{D}_1(u), \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_3(u)\mathcal{B}_2(v) &= b_{31}(u,v)\mathcal{B}_2(u)\mathcal{D}_3(v) + b_{32}(u,v)\mathcal{B}_2(u)\mathcal{D}_1(v) + b_{33}(u,v)\mathcal{B}_2(u)\mathcal{D}_2(v) \\
&+ b_{34}(u,v)\mathcal{B}_2(v)\mathcal{D}_3(u) + b_{35}(u,v)\mathcal{B}_1(u)\mathcal{B}_1(v) + b_{36}(u,v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\
&+ b_{37}(u,v)\mathcal{B}_3(u)\mathcal{B}_1(v) + b_{38}(u,v)\mathcal{B}_3(u)\mathcal{B}_3(v) + b_{39}(u,v)\mathcal{B}_2(v)\mathcal{D}_1(u) \\
&+ b_{310}(u,v)\mathcal{B}_2(v)\mathcal{D}_2(u), \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_1(u)\mathcal{B}_2(v) &= Y_{11}(u,v)\mathcal{B}_2(v)\mathcal{C}_1(u) + Y_{12}(u,v)\mathcal{B}_2(v)\mathcal{C}_3(u) + Y_{13}(u,v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
&+ Y_{14}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + Y_{15}(u,v)\mathcal{B}_1(v)\mathcal{D}_1(u) + Y_{16}(u,v)\mathcal{B}_1(v)\mathcal{D}_2(u) \\
&+ Y_{17}(u,v)\mathcal{B}_3(v)\mathcal{D}_1(u) + Y_{18}(u,v)\mathcal{B}_3(v)\mathcal{D}_2(u) + Y_{19}(u,v)\mathcal{B}_1(u)\mathcal{D}_1(v) \\
&+ Y_{110}(u,v)\mathcal{B}_1(u)\mathcal{D}_2(v) + Y_{111}(u,v)\mathcal{B}_1(u)\mathcal{D}_3(v) + Y_{112}(u,v)\mathcal{B}_3(u)\mathcal{D}_1(v) \\
&+ Y_{113}(u,v)\mathcal{B}_3(u)\mathcal{D}_2(v) + Y_{114}(u,v)\mathcal{B}_3(u)\mathcal{D}_3(v), \tag{A.12}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_2(u)\mathcal{B}_2(v) = & c_{31}(u,v)\mathcal{D}_2(u)\mathcal{D}_3(v) + c_{32}(u,v)\mathcal{B}_2(u)\mathcal{C}_2(v) + c_{33}(u,v)\mathcal{D}_3(u)\mathcal{D}_3(v) \\
& + c_{34}(u,v)\mathcal{D}_2(u)\mathcal{D}_2(v) + c_{35}(u,v)\mathcal{B}_3(u)\mathcal{C}_1(v) + c_{36}(u,v)\mathcal{D}_2(v)\mathcal{D}_3(u) \\
& + c_{37}(u,v)\mathcal{D}_1(u)\mathcal{D}_2(v) + c_{38}(u,v)\mathcal{D}_1(v)\mathcal{D}_3(u) + c_{39}(u,v)\mathcal{D}_1(v)\mathcal{D}_2(u) \\
& + c_{310}(u,v)\mathcal{B}_3(v)\mathcal{C}_1(u) + c_{311}(u,v)\mathcal{B}_1(u)\mathcal{C}_1(v) + c_{312}(u,v)\mathcal{D}_3(u)\mathcal{D}_1(v) \\
& + c_{313}(u,v)\mathcal{D}_3(v)\mathcal{D}_1(u) + c_{314}(u,v)\mathcal{D}_2(v)\mathcal{D}_2(u) + c_{315}(u,v)\mathcal{D}_3(u)\mathcal{D}_2(v) \\
& + c_{316}(u,v)\mathcal{D}_1(u)\mathcal{D}_3(v) + c_{317}(u,v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{318}(u,v)\mathcal{D}_1(v)\mathcal{D}_1(u) \\
& + c_{319}(u,v)\mathcal{D}_1(u)\mathcal{D}_1(v) + c_{320}(u,v)\mathcal{D}_2(u)\mathcal{D}_1(v) + c_{321}(u,v)\mathcal{B}_3(v)\mathcal{C}_3(u) \\
& + c_{322}(u,v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{323}(u,v)\mathcal{D}_2(v)\mathcal{D}_1(u) + c_{324}(u,v)\mathcal{B}_1(v)\mathcal{C}_3(u) \\
& + c_{325}(u,v)\mathcal{B}_3(u)\mathcal{C}_3(v), \tag{A.13}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_3(u)\mathcal{B}_2(v) = & Y_{21}(u,v)\mathcal{B}_2(v)\mathcal{C}_3(u) + Y_{22}(u,v)\mathcal{B}_2(v)\mathcal{C}_1(u) + Y_{23}(u,v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
& + Y_{24}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + Y_{25}(u,v)\mathcal{B}_1(v)\mathcal{D}_1(u) + Y_{26}(u,v)\mathcal{B}_1(v)\mathcal{D}_2(u) \\
& + Y_{27}(u,v)\mathcal{B}_1(v)\mathcal{D}_3(u) + Y_{28}(u,v)\mathcal{B}_3(v)\mathcal{D}_1(u) + Y_{29}(u,v)\mathcal{B}_3(v)\mathcal{D}_2(u) \\
& + Y_{210}(u,v)\mathcal{B}_3(v)\mathcal{D}_3(u) + Y_{211}(u,v)\mathcal{B}_1(u)\mathcal{D}_1(v) + Y_{212}(u,v)\mathcal{B}_1(u)\mathcal{D}_2(v) \\
& + Y_{213}(u,v)\mathcal{B}_1(u)\mathcal{D}_3(v) + Y_{214}(u,v)\mathcal{B}_3(u)\mathcal{D}_1(v) + Y_{215}(u,v)\mathcal{B}_3(u)\mathcal{D}_2(v) \\
& + Y_{216}(u,v)\mathcal{B}_3(u)\mathcal{D}_3(v), \tag{A.14}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_1(u)\mathcal{B}_3(v) = & x_{11}(u,v)\mathcal{B}_3(v)\mathcal{D}_1(u) + x_{12}(u,v)\mathcal{B}_1(v)\mathcal{D}_1(u) + x_{13}(u,v)\mathcal{B}_1(u)\mathcal{D}_1(v) \\
& + x_{14}(u,v)\mathcal{B}_1(u)\mathcal{D}_2(v) + x_{15}(u,v)\mathcal{B}_1(u)\mathcal{D}_3(v) + x_{16}(u,v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
& + x_{17}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + x_{18}(u,v)\mathcal{B}_2(v)\mathcal{C}_1(u), \tag{A.15}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_2(u)\mathcal{B}_3(v) = & x_{21}(u,v)\mathcal{B}_3(v)\mathcal{D}_2(u) + x_{22}(u,v)\mathcal{B}_1(v)\mathcal{D}_2(u) + x_{23}(u,v)\mathcal{B}_1(u)\mathcal{D}_1(v) \\
& + x_{24}(u,v)\mathcal{B}_1(u)\mathcal{D}_2(v) + x_{25}(u,v)\mathcal{B}_1(u)\mathcal{D}_3(v) + x_{26}(u,v)\mathcal{B}_3(u)\mathcal{D}_1(v) \\
& + x_{27}(u,v)\mathcal{B}_3(u)\mathcal{D}_2(v) + x_{28}(u,v)\mathcal{B}_3(u)\mathcal{D}_3(v) + x_{29}(u,v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
& + x_{210}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + x_{211}(u,v)\mathcal{B}_2(v)\mathcal{C}_1(u) + x_{212}(u,v)\mathcal{B}_2(v)\mathcal{C}_3(u), \tag{A.16}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_3(u)\mathcal{B}_3(v) = & x_{31}(u,v)\mathcal{B}_3(v)\mathcal{D}_3(u) + x_{32}(u,v)\mathcal{B}_1(v)\mathcal{D}_3(u) + x_{33}(u,v)\mathcal{B}_1(u)\mathcal{D}_1(v) \\
& + x_{34}(u,v)\mathcal{B}_1(u)\mathcal{D}_2(v) + x_{35}(u,v)\mathcal{B}_1(u)\mathcal{D}_3(v) + x_{36}(u,v)\mathcal{B}_3(u)\mathcal{D}_1(v) \\
& + x_{37}(u,v)\mathcal{B}_3(u)\mathcal{D}_2(v) + x_{38}(u,v)\mathcal{B}_3(u)\mathcal{D}_3(v) + x_{39}(u,v)\mathcal{B}_2(u)\mathcal{C}_1(v) \\
& + x_{310}(u,v)\mathcal{B}_2(u)\mathcal{C}_3(v) + x_{311}(u,v)\mathcal{B}_2(v)\mathcal{C}_1(u) + x_{312}(u,v)\mathcal{B}_2(v)\mathcal{C}_3(u) \\
& + x_{313}(u,v)\mathcal{B}_1(v)\mathcal{D}_1(u) + x_{314}(u,v)\mathcal{B}_3(v)\mathcal{D}_1(u), \tag{A.17}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_2(u)\mathcal{B}_1(v) = & e_{11}(u,v)\mathcal{B}_1(v)\mathcal{B}_2(u) + e_{12}(u,v)\mathcal{B}_2(v)\mathcal{B}_1(u) + e_{13}(u,v)\mathcal{B}_2(v)\mathcal{B}_3(u), \tag{A.18}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_1(u)\mathcal{B}_2(v) &= e_{21}(u, v)\mathcal{B}_2(v)\mathcal{B}_1(u) + e_{22}(u, v)\mathcal{B}_2(v)\mathcal{B}_3(u) + e_{23}(u, v)\mathcal{B}_1(v)\mathcal{B}_2(u) \\
&+ e_{24}(u, v)\mathcal{B}_3(v)\mathcal{B}_2(u),
\end{aligned} \tag{A.19}$$

$$\begin{aligned}
\mathcal{B}_2(u)\mathcal{B}_3(v) &= e_{31}(u, v)\mathcal{B}_3(v)\mathcal{B}_2(u) + e_{32}(u, v)\mathcal{B}_1(v)\mathcal{B}_2(u) + e_{33}(u, v)\mathcal{B}_2(v)\mathcal{B}_1(u) \\
&+ e_{34}(u, v)\mathcal{B}_2(v)\mathcal{B}_3(u),
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
\mathcal{C}_1(u)\mathcal{B}_3(v) &= C_{21}(u, v)\mathcal{B}_1(v)\mathcal{C}_1(u) + C_{22}(u, v)\mathcal{B}_3(v)\mathcal{C}_1(u) + C_{23}(u, v)\mathcal{B}_3(u)\mathcal{C}_1(v) \\
&+ C_{24}(u, v)\mathcal{B}_3(u)\mathcal{C}_3(v) + C_{25}(u, v)\mathcal{B}_2(v)\mathcal{C}_2(u) + C_{26}(u, v)\mathcal{D}_1(v)\mathcal{D}_1(u) \\
&+ C_{27}(u, v)\mathcal{D}_2(v)\mathcal{D}_1(u) + C_{28}(u, v)\mathcal{D}_3(v)\mathcal{D}_1(u) + C_{29}(u, v)\mathcal{D}_1(u)\mathcal{D}_1(v) \\
&+ C_{210}(u, v)\mathcal{D}_2(u)\mathcal{D}_1(v) + C_{211}(u, v)\mathcal{D}_1(u)\mathcal{D}_2(v) + C_{212}(u, v)\mathcal{D}_2(u)\mathcal{D}_2(v) \\
&+ C_{213}(u, v)\mathcal{D}_1(u)\mathcal{D}_3(v) + C_{214}(u, v)\mathcal{D}_2(u)\mathcal{D}_3(v),
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
\mathcal{C}_3(u)\mathcal{B}_3(v) &= c_{41}(u, v)\mathcal{D}_1(v)\mathcal{D}_2(u) + c_{42}(u, v)\mathcal{D}_1(u)\mathcal{D}_2(v) + c_{43}(u, v)\mathcal{D}_2(u)\mathcal{D}_1(v) \\
&+ c_{44}(u, v)\mathcal{B}_1(u)\mathcal{C}_1(v) + c_{45}(u, v)\mathcal{D}_2(v)\mathcal{D}_3(u) + c_{46}(u, v)\mathcal{B}_3(v)\mathcal{C}_1(u) \\
&+ c_{47}(u, v)\mathcal{B}_2(v)\mathcal{C}_2(u) + c_{48}(u, v)\mathcal{D}_3(u)\mathcal{D}_1(v) + c_{49}(u, v)\mathcal{D}_2(v)\mathcal{D}_1(u) \\
&+ c_{410}(u, v)\mathcal{D}_3(u)\mathcal{D}_2(v) + c_{411}(u, v)\mathcal{B}_1(v)\mathcal{C}_3(u) + c_{412}(u, v)\mathcal{D}_2(v)\mathcal{D}_2(u) \\
&+ c_{413}(u, v)\mathcal{B}_1(v)\mathcal{C}_1(u) + c_{414}(u, v)\mathcal{D}_1(u)\mathcal{D}_3(v) + c_{415}(u, v)\mathcal{D}_1(v)\mathcal{D}_3(u) \\
&+ c_{416}(u, v)\mathcal{D}_2(u)\mathcal{D}_2(v) + c_{417}(u, v)\mathcal{D}_1(v)\mathcal{D}_1(u) + c_{418}(u, v)\mathcal{D}_1(u)\mathcal{D}_1(v) \\
&+ c_{419}(u, v)\mathcal{B}_2(u)\mathcal{C}_2(v) + c_{420}(u, v)\mathcal{D}_3(v)\mathcal{D}_1(u) + c_{421}(u, v)\mathcal{B}_3(u)\mathcal{C}_3(v) \\
&+ c_{422}(u, v)\mathcal{B}_3(v)\mathcal{C}_3(u) + c_{423}(u, v)\mathcal{B}_3(u)\mathcal{C}_1(v) + c_{424}(u, v)\mathcal{D}_3(u)\mathcal{D}_3(v) \\
&+ c_{425}(u, v)\mathcal{D}_2(u)\mathcal{D}_3(v),
\end{aligned} \tag{A.22}$$

$$\begin{aligned}
\mathcal{B}_3(u)\mathcal{B}_1(v) &= d_{21}(u, v)\mathcal{B}_1(v)\mathcal{B}_3(u) + d_{22}(u, v)\mathcal{B}_1(v)\mathcal{B}_1(u) + d_{23}(u, v)\mathcal{B}_2(v)\mathcal{D}_1(u) \\
&+ d_{24}(u, v)\mathcal{B}_2(v)\mathcal{D}_2(u) + d_{25}(u, v)\mathcal{B}_2(v)\mathcal{D}_3(u) + d_{26}(u, v)\mathcal{B}_2(u)\mathcal{D}_1(v) \\
&+ d_{27}(u, v)\mathcal{B}_2(u)\mathcal{D}_2(v),
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
\mathcal{B}_3(u)\mathcal{B}_3(v) &= f_{01}(u, v)\mathcal{B}_3(v)\mathcal{B}_3(u) + f_{02}(u, v)\mathcal{B}_1(u)\mathcal{B}_1(v) + f_{03}(u, v)\mathcal{B}_1(u)\mathcal{B}_3(v) \\
&+ f_{04}(u, v)\mathcal{B}_3(u)\mathcal{B}_1(v) + f_{05}(u, v)\mathcal{B}_2(u)\mathcal{D}_1(v) + f_{06}(u, v)\mathcal{B}_2(v)\mathcal{D}_1(u) \\
&+ f_{07}(u, v)\mathcal{B}_2(u)\mathcal{D}_2(v) + f_{08}(u, v)\mathcal{B}_2(v)\mathcal{D}_2(u) + f_{09}(u, v)\mathcal{B}_2(u)\mathcal{D}_3(v) \\
&+ f_{010}(u, v)\mathcal{B}_2(v)\mathcal{D}_3(u).
\end{aligned} \tag{A.24}$$

Here we write explicitly only the amplitudes which appear in the final solution (subsection 3.4). For more details, including explicit expressions for the coefficients of the commutation relations, see for instance [7]. For the ZF model we have,

$$a_{11}(u, v) = \frac{\sinh(u + v) \sinh(u - v - \eta)}{\sinh(u - v) \sinh(u + v + \eta)}, \tag{A.25}$$

$$a_{21}(u, v) = \frac{\sinh(u+v) \sinh(u-v-\eta) \sinh(u-v+\frac{\eta}{2}) \sinh(u+v+\frac{3\eta}{2})}{\sinh(u-v) \sinh(u-v-\frac{\eta}{2}) \sinh(u+v+\frac{\eta}{2}) \sinh(u+v+\eta)}, \quad (\text{A.26})$$

$$a_{31}(u, v) = \frac{\sinh(u-v+\frac{\eta}{2}) \sinh(u+v+\frac{3\eta}{2})}{\sinh(u-v-\frac{\eta}{2}) \sinh(u+v+\frac{\eta}{2})}, \quad (\text{A.27})$$

$$e_{01}(u, v) = \frac{\sinh(u-v-\eta) \sinh(u-v+\frac{\eta}{2})}{\sinh(u-v-\frac{\eta}{2}) \sinh(u-v+\eta)}, \quad (\text{A.28})$$

$$e_{04}(u, v) = \frac{\sinh(2v) \sinh(\eta)}{\sinh(u-v-\frac{\eta}{2}) \sinh(2v+\eta)}, \quad e_{05}(u, v) = -\frac{\sinh(\eta)}{\sinh(u+v+\frac{\eta}{2})}, \quad (\text{A.29})$$

and, for the IK solution, we have,

$$a_{11}(u, v) = \frac{\sinh(u+v) \sinh(u-v-\eta)}{\sinh(u-v) \sinh(u+v+\eta)}, \quad (\text{A.30})$$

$$a_{21}(u, v) = \frac{\sinh(u-v+\eta) \sinh(u+v+2\eta) \cosh(u-v-\frac{\eta}{2}) \cosh(u+v+\frac{\eta}{2})}{\sinh(u-v) \sinh(u+v+\eta) \cosh(u-v+\frac{\eta}{2}) \cosh(u+v+\frac{3\eta}{2})}, \quad (\text{A.31})$$

$$a_{31}(u, v) = \frac{\cosh(u-v+\frac{3\eta}{2}) \cosh(u+v+\frac{5\eta}{2})}{\cosh(u-v+\frac{\eta}{2}) \cosh(u+v+\frac{3\eta}{2})}, \quad (\text{A.32})$$

$$e_{01}(u, v) = \frac{\cosh(u-v-\frac{\eta}{2})}{\cosh(u-v+\frac{\eta}{2})}, \quad (\text{A.33})$$

$$e_{04}(u, v) = \frac{e^\eta \sinh(2v) \sinh(\eta)}{\cosh(u-v+\frac{\eta}{2}) \sinh(2v+\eta)}, \quad e_{05}(u, v) = -\frac{e^\eta \sinh(\eta)}{\cosh(u+v+\frac{3\eta}{2})}. \quad (\text{A.34})$$

B Coefficients of the expansions $t_d(u)\Psi_n$ and $t_u(u)\Psi_n$

The functions entering equation (37) are given by,

$$\Lambda_n(u, u_1, \dots, u_n) = \sum_{\alpha=1}^3 \omega_\alpha(u) \Delta_\alpha(u) \prod_{j=1}^n a_{\alpha 1}(u, u_j), \quad (\text{B.1})$$

$$\mathcal{F}_j^{(n)}(u, u_1, \dots, u_n) = \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \sum_{\alpha=1}^2 \Delta_\alpha(u_j) Q_\alpha^{\mathcal{F}}(u, u_j) \prod_{m=1, m \neq j}^n a_{\alpha 1}(u_j, u_m) \right\}, \quad (\text{B.2})$$

$$\mathcal{G}_j^{(n)}(u, u_1, \dots, u_n) = \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \sum_{\alpha=1}^2 \Delta_\alpha(u_j) Q_\alpha^{\mathcal{G}}(u, u_j) \prod_{m=1, m \neq j}^n a_{\alpha 1}(u_j, u_m) \right\}, \quad (\text{B.3})$$

and

$$\begin{aligned} \mathcal{H}_{jk}^{(n)}(u, u_1, \dots, u_n) &= \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \\ &\times \left\{ \sum_{\alpha, \beta=1}^2 \Delta_\alpha(u_j) \Delta_\beta(u_k) Q_{\alpha\beta}^{\mathcal{H}}(u, u_j, u_k) \prod_{\substack{m=1 \\ m \neq j, k}}^n a_{\alpha 1}(u_j, u_m) a_{\beta 1}(u_k, u_m) \right\} \end{aligned} \quad (\text{B.4})$$

with the auxiliary functions Q defined by,

$$Q_1^{\mathcal{F}}(u, u_j) = \sum_{q=1}^3 \omega_q(u) a_{q2}(u, u_j), \quad Q_2^{\mathcal{F}}(u, u_j) = \sum_{q=1}^3 \omega_q(u) a_{q3}(u, u_j), \quad (\text{B.5})$$

$$Q_1^{\mathcal{G}}(u, u_j) = \sum_{q=2}^3 \omega_q(u) a_{q4}(u, u_j), \quad Q_2^{\mathcal{G}}(u, u_j) = \sum_{q=2}^3 \omega_q(u) a_{q5}(u, u_j), \quad (\text{B.6})$$

$$\begin{aligned} Q_{11}^{\mathcal{H}}(u, u_j, u_k) &= \omega_1(u) \{ a_{11}(u, u_j) a_{12}(u, u_k) e_{03}(u_j, u) + a_{14}(u, u_j) [c_{16}(u_j, u_k) \\ &+ c_{18}(u_j, u_k)] + a_{15}(u, u_j) [c_{26}(u_j, u_k) + c_{29}(u_j, u_k)] \\ &- b_{12}(u, u_j) e_{04}(u_j, u_k) \} \\ &+ \omega_2(u) \{ a_{21}(u, u_j) [a_{22}(u, u_k) e_{03}(u_j, u) + a_{24}(u, u_k) d_{13}(u_j, u)] \\ &+ a_{26}(u, u_j) [c_{16}(u_j, u_k) + c_{18}(u_j, u_k)] + a_{27}(u, u_j) [c_{26}(u_j, u_k) \\ &+ c_{29}(u_j, u_k)] - b_{22}(u, u_j) e_{04}(u_j, u_k) \} \\ &+ \omega_3(u) \{ a_{31}(u, u_j) [a_{32}(u, u_k) e_{03}(u_j, u) + a_{34}(u, u_k) d_{13}(u_j, u)] \\ &+ a_{36}(u, u_j) [c_{16}(u_j, u_k) + c_{18}(u_j, u_k)] + a_{37}(u, u_j) [c_{26}(u_j, u_k) \\ &+ c_{29}(u_j, u_k)] - b_{32}(u, u_j) e_{04}(u_j, u_k) \}, \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} Q_{12}^{\mathcal{H}}(u, u_j, u_k) &= \omega_1(u) \{ a_{11}(u, u_j) a_{13}(u, u_k) e_{03}(u_j, u) + a_{14}(u, u_j) c_{19}(u_j, u_k) \\ &+ a_{15}(u, u_j) c_{210}(u_j, u_k) - b_{12}(u, u_j) e_{05}(u_j, u_k) \} \\ &+ \omega_2(u) \{ a_{21}(u, u_j) [a_{23}(u, u_k) e_{03}(u_j, u) + a_{25}(u, u_k) d_{13}(u_j, u)] \\ &+ a_{26}(u, u_j) c_{19}(u_j, u_k) + a_{27}(u, u_j) c_{210}(u_j, u_k) - b_{22}(u, u_j) e_{05}(u_j, u_k) \} \\ &+ \omega_3(u) \{ a_{31}(u, u_j) [a_{33}(u, u_k) e_{03}(u_j, u) + a_{35}(u, u_k) d_{13}(u_j, u)] \\ &+ a_{36}(u, u_j) c_{19}(u_j, u_k) + a_{37}(u, u_j) c_{210}(u_j, u_k) - b_{32}(u, u_j) e_{05}(u_j, u_k) \}, \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned}
Q_{21}^{\mathcal{H}}(u, u_j, u_k) = & \omega_1(u) \{a_{11}(u, u_j)a_{12}(u, u_k)e_{02}(u_j, u) + a_{14}(u, u_j)[c_{17}(u_j, u_k) \\
& + c_{110}(u_j, u_k)] + a_{15}(u, u_j)[c_{27}(u_j, u_k) + c_{211}(u_j, u_k)] \\
& - b_{13}(u, u_j)e_{04}(u_j, u_k)\} \\
& + \omega_2(u) \{a_{21}(u, u_j)[a_{22}(u, u_k)e_{02}(u_j, u) + a_{24}(u, u_k)d_{14}(u_j, u)] \\
& + a_{26}(u, u_j)[c_{17}(u_j, u_k) + c_{110}(u_j, u_k)] + a_{27}(u, u_j)[c_{27}(u_j, u_k) \\
& + c_{211}(u_j, u_k)] - b_{23}(u, u_j)e_{04}(u_j, u_k)\} \\
& + \omega_3(u) \{a_{31}(u, u_j)[a_{32}(u, u_k)e_{02}(u_j, u) + a_{34}(u, u_k)d_{14}(u_j, u)] \\
& + a_{36}(u, u_j)[c_{17}(u_j, u_k) + c_{110}(u_j, u_k)] + a_{37}(u, u_j)[c_{27}(u_j, u_k) \\
& + c_{211}(u_j, u_k)] - b_{33}(u, u_j)e_{04}(u_j, u_k)\}, \tag{B.9}
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{\mathcal{H}}(u, u_j, u_k) = & \omega_1(u) \{a_{11}(u, u_j)a_{13}(u, u_k)e_{02}(u_j, u) + a_{14}(u, u_j)c_{111}(u_j, u_k) \\
& + a_{15}(u, u_j)c_{212}(u_j, u_k) - b_{13}(u, u_j)e_{05}(u_j, u_k)\} \\
& + \omega_2(u) \{a_{21}(u, u_j)[a_{23}(u, u_k)e_{02}(u_j, u) + a_{25}(u, u_k)d_{14}(u_j, u)] \\
& + a_{26}(u, u_j)c_{111}(u_j, u_k) + a_{27}(u, u_j)c_{212}(u_j, u_k) - b_{23}(u, u_j)e_{05}(u_j, u_k)\} \\
& + \omega_3(u) \{a_{31}(u, u_j)[a_{33}(u, u_k)e_{02}(u_j, u) + a_{35}(u, u_k)d_{14}(u_j, u)] \\
& + a_{36}(u, u_j)c_{111}(u_j, u_k) + a_{37}(u, u_j)c_{212}(u_j, u_k) - b_{33}(u, u_j)e_{05}(u_j, u_k)\}. \tag{B.10}
\end{aligned}$$

For the expansion (38) we have,

$$\begin{aligned}
\mathcal{T}_j^{(n)}(u, u_1, \dots, u_n) = & \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \\
& \times \left\{ \sum_{\alpha=1}^3 \sum_{\beta=1}^2 \Delta_{\alpha}(u) \Delta_{\beta}(u_j) Q_{\alpha\beta}^{\mathcal{T}}(u, u_j) \prod_{\substack{m=1 \\ m \neq j}}^n a_{\alpha 1}(u, u_m) a_{\beta 1}(u_j, u_m) \right\}, \tag{B.11}
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}_{jk}^{(n)}(u, u_1, \dots, u_n) = & \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \\
& \times \left\{ \sum_{\alpha=1}^3 \sum_{\substack{\beta=1 \\ \gamma=1}}^2 \Delta_{\alpha}(u) \Delta_{\beta}(u_j) \Delta_{\gamma}(u_k) Q_{\alpha\beta\gamma}^{\mathcal{U}}(u, u_j, u_k) \prod_{\substack{m=1 \\ m \neq j, k}}^n a_{\alpha 1}(u, u_m) a_{\beta 1}(u_j, u_m) a_{\gamma 1}(u_k, u_m) \right\} \tag{B.12}
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{jk}^{(n)}(u, u_1, \dots, u_n) &= \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \\
&\times \left\{ \sum_{\alpha, \beta=1}^2 \Delta_\alpha(u_j) \Delta_\beta(u_k) Q_{\alpha\beta}^\mathcal{V}(u, u_j, u_k) \prod_{\substack{m=1 \\ m \neq j, k}}^n a_{\alpha 1}(u_j, u_m) a_{\beta 1}(u_k, u_m) \right\}
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
\mathcal{W}_{jk}^{(n)}(u, u_1, \dots, u_n) &= \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \\
&\times \left\{ \sum_{\alpha, \beta=1}^2 \Delta_\alpha(u_j) \Delta_\beta(u_k) Q_{\alpha\beta}^\mathcal{W}(u, u_j, u_k) \prod_{\substack{m=1 \\ m \neq j, k}}^n a_{\alpha 1}(u_j, u_m) a_{\beta 1}(u_k, u_m) \right\}
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\mathcal{X}_{jk\ell}^{(n)}(u, u_1, \dots, u_n) &= \\
&\left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \left\{ \prod_{r=1, r < \ell, r \neq j, k}^n \Omega(u_\ell, u_r) \right\} \\
&\times \left\{ \sum_{\alpha, \beta, \gamma=1}^2 \Delta_\alpha(u_j) \Delta_\beta(u_k) \Delta_\gamma(u_\ell) Q_{\alpha\beta\gamma}^\mathcal{X}(u, u_j, u_k, u_\ell) \times \right. \\
&\quad \left. \prod_{m=1, m \neq j, k, \ell}^n a_{\alpha 1}(u_j, u_m) a_{\beta 1}(u_k, u_m) a_{\gamma 1}(u_\ell, u_m) \right\},
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\mathcal{Y}_{jk\ell}^{(n)}(u, u_1, \dots, u_n) &= \\
&\left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \left\{ \prod_{r=1, r < \ell, r \neq j, k}^n \Omega(u_\ell, u_r) \right\} \\
&\times \left\{ \sum_{\alpha, \beta, \gamma=1}^2 \Delta_\alpha(u_j) \Delta_\beta(u_k) \Delta_\gamma(u_\ell) Q_{\alpha\beta\gamma}^\mathcal{Y}(u, u_j, u_k, u_\ell) \times \right. \\
&\quad \left. \prod_{m=1, m \neq j, k, \ell}^n a_{\alpha 1}(u_j, u_m) a_{\beta 1}(u_k, u_m) a_{\gamma 1}(u_\ell, u_m) \right\},
\end{aligned} \tag{B.16}$$

$$\begin{aligned}
& \mathcal{Z}_{jk\ell}^{(n)}(u, u_1, \dots, u_n) = \\
& \left\{ \prod_{p=1, p < j}^n \Omega(u_j, u_p) \right\} \left\{ \prod_{q=1, q < k, q \neq j}^n \Omega(u_k, u_q) \right\} \left\{ \prod_{r=1, r < \ell, r \neq j, k}^n \Omega(u_\ell, u_r) \right\} \\
& \times \left\{ \sum_{\alpha, \beta, \gamma=1}^2 \Delta_\alpha(u_j) \Delta_\beta(u_k) \Delta_\gamma(u_\ell) Q_{\alpha\beta\gamma}^{\mathcal{Z}}(u, u_j, u_k, u_\ell) \times \right. \\
& \left. \prod_{m=1, m \neq j, k, \ell}^n a_{\alpha 1}(u_j, u_m) a_{\beta 1}(u_k, u_m) a_{\gamma 1}(u_\ell, u_m) \right\}.
\end{aligned} \tag{B.17}$$

We have the following expressions,

$$Q_{11}^{\mathcal{T}}(u, u_j) = k_{12}^+(u)[c_{16}(u, u_j) + c_{18}(u, u_j)] + k_{23}^+(u)[c_{26}(u, u_j) + c_{29}(u, u_j)], \tag{B.18}$$

$$Q_{12}^{\mathcal{T}}(u, u_j) = k_{12}^+(u)c_{19}(u, u_j) + k_{23}^+(u)c_{210}(u, u_j), \tag{B.19}$$

$$Q_{21}^{\mathcal{T}}(u, u_j) = k_{12}^+(u)[c_{110}(u, u_j) + c_{17}(u, u_j)] + k_{23}^+(u)[c_{211}(u, u_j) + c_{27}(u, u_j)], \tag{B.20}$$

$$Q_{22}^{\mathcal{T}}(u, u_j) = k_{12}^+(u)c_{111}(u, u_j) + k_{23}^+(u)c_{212}(u, u_j), \tag{B.21}$$

$$Q_{31}^{\mathcal{T}}(u, u_j) = k_{23}^+(u)[c_{213}(u, u_j) + c_{28}(u, u_j)], \quad Q_{32}^{\mathcal{T}}(u, u_j) = k_{23}^+(u)c_{214}(u, u_j), \tag{B.22}$$

$$\begin{aligned}
& Q_{111}^{\mathcal{U}}(u, u_j, u_k) = \\
& k_{13}^+(u) [y_{15}(u, u_j) (c_{16}(u, u_k) + c_{18}(u, u_k)) + y_{19}(u, u_j) (c_{16}(u_j, u_k) + c_{18}(u_j, u_k)) \\
& + y_{112}(u, u_j) (c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) + y_{17}(u, u_j) (c_{26}(u, u_k) + c_{29}(u, u_k)) \\
& - e_{04}(u_j, u_k) (c_{318}(u, u_j) + c_{319}(u, u_j))],
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
& Q_{112}^{\mathcal{U}}(u, u_j, u_k) = k_{13}^+(u) [c_{19}(u, u_k) y_{15}(u, u_j) + y_{19}(u, u_j) c_{19}(u_j, u_k) \\
& + y_{112}(u, u_j) c_{210}(u_j, u_k) + c_{210}(u, u_k) y_{17}(u, u_j) \\
& - e_{05}(u_j, u_k) (c_{318}(u, u_j) + c_{319}(u, u_j))],
\end{aligned} \tag{B.24}$$

$$\begin{aligned}
& Q_{121}^{\mathcal{U}}(u, u_j, u_k) = \\
& k_{13}^+(u) [y_{19}(u, u_j) (c_{110}(u_j, u_k) + c_{17}(u_j, u_k)) + y_{16}(u, u_j) (c_{16}(u, u_k) + c_{18}(u, u_k)) \\
& + y_{112}(u, u_j) (c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) + y_{18}(u, u_j) (c_{26}(u, u_k) + c_{29}(u, u_k)) \\
& - e_{04}(u_j, u_k) (c_{323}(u, u_j) + c_{37}(u, u_j))],
\end{aligned} \tag{B.25}$$

$$\begin{aligned}
& Q_{122}^{\mathcal{U}}(u, u_j, u_k) = k_{13}^+(u) [y_{19}(u, u_j) c_{111}(u_j, u_k) + c_{19}(u, u_k) y_{16}(u, u_j) \\
& + c_{210}(u, u_k) y_{18}(u, u_j) + y_{112}(u, u_j) c_{212}(u_j, u_k) \\
& - e_{05}(u_j, u_k) (c_{323}(u, u_j) + c_{37}(u, u_j))],
\end{aligned} \tag{B.26}$$

$$\begin{aligned}
Q_{211}^{\mathcal{U}}(u, u_j, u_k) = & \\
& k_{13}^+(u) [y_{15}(u, u_j) (c_{110}(u, u_k) + c_{17}(u, u_k)) + y_{110}(u, u_j) (c_{16}(u_j, u_k) + c_{18}(u_j, u_k)) \\
& + y_{17}(u, u_j) (c_{211}(u, u_k) + c_{27}(u, u_k)) + y_{113}(u, u_j) (c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) \\
& - e_{04}(u_j, u_k) (c_{320}(u, u_j) + c_{39}(u, u_j))] , \tag{B.27}
\end{aligned}$$

$$\begin{aligned}
Q_{212}^{\mathcal{U}}(u, u_j, u_k) = & k_{13}^+(u) [c_{111}(u, u_k) y_{15}(u, u_j) + y_{110}(u, u_j) c_{19}(u_j, u_k) \\
& + y_{113}(u, u_j) c_{210}(u_j, u_k) + c_{212}(u, u_k) y_{17}(u, u_j) \\
& - e_{05}(u_j, u_k) (c_{320}(u, u_j) + c_{39}(u, u_j))] , \tag{B.28}
\end{aligned}$$

$$\begin{aligned}
Q_{221}^{\mathcal{U}}(u, u_j, u_k) = & \\
& k_{13}^+(u) [y_{110}(u, u_j) (c_{110}(u_j, u_k) + c_{17}(u_j, u_k)) + y_{16}(u, u_j) (c_{110}(u, u_k) + c_{17}(u, u_k)) \\
& + y_{113}(u, u_j) (c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) + y_{18}(u, u_j) (c_{211}(u, u_k) + c_{27}(u, u_k)) \\
& - e_{04}(u_j, u_k) (c_{314}(u, u_j) + c_{34}(u, u_j))] , \tag{B.29}
\end{aligned}$$

$$\begin{aligned}
Q_{222}^{\mathcal{U}}(u, u_j, u_k) = & k_{13}^+(u) [y_{110}(u, u_j) c_{111}(u_j, u_k) + c_{111}(u, u_k) y_{16}(u, u_j) \\
& + y_{113}(u, u_j) c_{212}(u_j, u_k) + c_{212}(u, u_k) y_{18}(u, u_j) \\
& - e_{05}(u_j, u_k) (c_{314}(u, u_j) + c_{34}(u, u_j))] , \tag{B.30}
\end{aligned}$$

$$\begin{aligned}
Q_{311}^{\mathcal{U}}(u, u_j, u_k) = & \\
& k_{13}^+(u) [y_{111}(u, u_j) (c_{16}(u_j, u_k) + c_{18}(u_j, u_k)) + y_{17}(u, u_j) (c_{213}(u, u_k) + c_{28}(u, u_k)) \\
& + y_{114}(u, u_j) (c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) - e_{04}(u_j, u_k) (c_{312}(u, u_j) + c_{38}(u, u_j))] , \tag{B.31}
\end{aligned}$$

$$\begin{aligned}
Q_{312}^{\mathcal{U}}(u, u_j, u_k) = & k_{13}^+(u) [y_{111}(u, u_j) c_{19}(u_j, u_k) + y_{114}(u, u_j) c_{210}(u_j, u_k) \\
& + c_{214}(u, u_k) y_{17}(u, u_j) - e_{05}(u_j, u_k) (c_{312}(u, u_j) + c_{38}(u, u_j))] , \tag{B.32}
\end{aligned}$$

$$\begin{aligned}
Q_{321}^{\mathcal{U}}(u, u_j, u_k) = & k_{13}^+(u) [y_{111}(u, u_j) (c_{110}(u_j, u_k) + c_{17}(u_j, u_k)) \\
& + y_{114}(u, u_j) (c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) + y_{18}(u, u_j) (c_{213}(u, u_k) + c_{28}(u, u_k)) \\
& - e_{04}(u_j, u_k) (c_{315}(u, u_j) + c_{36}(u, u_j))] , \tag{B.33}
\end{aligned}$$

$$\begin{aligned}
Q_{322}^{\mathcal{U}}(u, u_j, u_k) = & k_{13}^+(u) [y_{111}(u, u_j) c_{111}(u_j, u_k) + y_{114}(u, u_j) c_{212}(u_j, u_k) \\
& + c_{214}(u, u_k) y_{18}(u, u_j) - e_{05}(u_j, u_k) (c_{315}(u, u_j) + c_{36}(u, u_j))] , \tag{B.34}
\end{aligned}$$

$$\begin{aligned}
Q_{11}^{\mathcal{V}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{11}(u_j, u_k)(a_{12}(u, u_k)c_{29}(u, u_j) + a_{22}(u, u_k)c_{211}(u, u_j) + a_{32}(u, u_k)c_{213}(u, u_j)) \\
&+ a_{11}(u_j, u)(a_{12}(u, u_k)c_{26}(u, u_j) + a_{22}(u, u_k)c_{27}(u, u_j) + a_{32}(u, u_k)c_{28}(u, u_j)) \\
&+ a_{12}(u_j, u_k)(a_{22}(u, u_j)c_{211}(u, u_j) + a_{32}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{12}(u, u_j)(c_{29}(u, u_j)a_{12}(u_j, u_k) + c_{210}(u, u_j)a_{22}(u_j, u_k)) \\
&+ c_{214}(u, u_j)(a_{32}(u, u_j)a_{22}(u_j, u_k) + x_{33}(u, u_j)a_{24}(u_j, u_k)) \\
&+ a_{22}(u, u_j)c_{212}(u, u_j)a_{22}(u_j, u_k) \\
&+ x_{12}(u_j, u)(a_{24}(u, u_k)c_{27}(u, u_j) + a_{34}(u, u_k)c_{28}(u, u_j)) \\
&+ a_{24}(u_j, u_k)(c_{210}(u, u_j)x_{13}(u, u_j) + c_{212}(u, u_j)x_{23}(u, u_j)) \\
&+ c_{23}(u, u_j)(c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) - Y_{211}(u, u_j)e_{04}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{11}(u_j, u_k)(a_{12}(u, u_k)c_{18}(u, u_j) + a_{22}(u, u_k)c_{110}(u, u_j)) \\
&+ a_{11}(u_j, u)a_{12}(u, u_k)c_{16}(u, u_j) \\
&+ c_{17}(u, u_j)(a_{11}(u_j, u)a_{22}(u, u_k) + a_{24}(u, u_k)x_{12}(u_j, u)) \\
&+ c_{19}(u, u_j)(a_{12}(u, u_j)a_{22}(u_j, u_k) + x_{13}(u, u_j)a_{24}(u_j, u_k)) \\
&+ a_{12}(u_j, u_k)(a_{12}(u, u_j)c_{18}(u, u_j) + a_{22}(u, u_j)c_{110}(u, u_j)) \\
&+ a_{22}(u, u_j)c_{111}(u, u_j)a_{22}(u_j, u_k) + c_{111}(u, u_j)x_{23}(u, u_j)a_{24}(u_j, u_k) \\
&+ c_{13}(u, u_j)(c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) - Y_{19}(u, u_j)e_{04}(u_j, u_k)], \tag{B.35}
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{\mathcal{V}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{11}(u_j, u_k)(a_{13}(u, u_k)c_{29}(u, u_j) + a_{23}(u, u_k)c_{211}(u, u_j) + a_{33}(u, u_k)c_{213}(u, u_j)) \\
&+ a_{11}(u_j, u)(a_{13}(u, u_k)c_{26}(u, u_j) + a_{23}(u, u_k)c_{27}(u, u_j) + a_{33}(u, u_k)c_{28}(u, u_j)) \\
&+ a_{12}(u, u_j)(c_{29}(u, u_j)a_{13}(u_j, u_k) + c_{210}(u, u_j)a_{23}(u_j, u_k)) \\
&+ a_{13}(u_j, u_k)(a_{22}(u, u_j)c_{211}(u, u_j) + a_{32}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{22}(u, u_j)c_{212}(u, u_j)a_{23}(u_j, u_k) \\
&+ c_{214}(u, u_j)(a_{32}(u, u_j)a_{23}(u_j, u_k) + x_{33}(u, u_j)a_{25}(u_j, u_k)) \\
&+ x_{12}(u_j, u)(a_{25}(u, u_k)c_{27}(u, u_j) + a_{35}(u, u_k)c_{28}(u, u_j)) \\
&+ a_{25}(u_j, u_k)(c_{210}(u, u_j)x_{13}(u, u_j) + c_{212}(u, u_j)x_{23}(u, u_j)) + c_{23}(u, u_j)c_{210}(u_j, u_k) \\
&- Y_{211}(u, u_j)e_{05}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{11}(u_j, u_k)(a_{13}(u, u_k)c_{18}(u, u_j) + a_{23}(u, u_k)c_{110}(u, u_j)) \\
&+ a_{11}(u_j, u)a_{13}(u, u_k)c_{16}(u, u_j) \\
&+ c_{17}(u, u_j)(a_{11}(u_j, u)a_{23}(u, u_k) + a_{25}(u, u_k)x_{12}(u_j, u)) \\
&+ a_{13}(u_j, u_k)(a_{12}(u, u_j)c_{18}(u, u_j) + a_{22}(u, u_j)c_{110}(u, u_j)) \\
&+ c_{19}(u, u_j)(a_{12}(u, u_j)a_{23}(u_j, u_k) + x_{13}(u, u_j)a_{25}(u_j, u_k)) \\
&+ a_{22}(u, u_j)c_{111}(u, u_j)a_{23}(u_j, u_k) + c_{111}(u, u_j)x_{23}(u, u_j)a_{25}(u_j, u_k) \\
&+ c_{13}(u, u_j)c_{210}(u_j, u_k) - Y_{19}(u, u_j)e_{05}(u_j, u_k)], \tag{B.36}
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{\mathcal{V}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{13}(u, u_j)(c_{29}(u, u_j)a_{12}(u_j, u_k) + c_{210}(u, u_j)a_{22}(u_j, u_k)) \\
&+ a_{12}(u, u_k)c_{210}(u, u_j)a_{21}(u_j, u_k) \\
&+ a_{12}(u_j, u_k)(a_{23}(u, u_j)c_{211}(u, u_j) + a_{33}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{21}(u_j, u_k)(a_{22}(u, u_k)c_{212}(u, u_j) + a_{32}(u, u_k)c_{214}(u, u_j)) \\
&+ a_{23}(u, u_j)c_{212}(u, u_j)a_{22}(u_j, u_k) + a_{33}(u, u_j)c_{214}(u, u_j)a_{22}(u_j, u_k) \\
&+ c_{210}(u, u_j)x_{14}(u, u_j)a_{24}(u_j, u_k) + c_{212}(u, u_j)x_{24}(u, u_j)a_{24}(u_j, u_k) \\
&+ c_{214}(u, u_j)x_{34}(u, u_j)a_{24}(u_j, u_k) + c_{23}(u, u_j)(c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) \\
&- Y_{212}(u, u_j)e_{04}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{12}(u_j, u_k)(a_{13}(u, u_j)c_{18}(u, u_j) + a_{23}(u, u_j)c_{110}(u, u_j)) \\
&+ a_{21}(u_j, u_k)(a_{12}(u, u_k)c_{19}(u, u_j) + a_{22}(u, u_k)c_{111}(u, u_j)) \\
&+ a_{22}(u_j, u_k)(a_{13}(u, u_j)c_{19}(u, u_j) + a_{23}(u, u_j)c_{111}(u, u_j)) \\
&+ a_{24}(u_j, u_k)(c_{111}(u, u_j)x_{24}(u, u_j) + c_{19}(u, u_j)x_{14}(u, u_j)) \\
&+ c_{13}(u, u_j)(c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) - Y_{110}(u, u_j)e_{04}(u_j, u_k)], \tag{B.37}
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{\mathcal{V}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{13}(u_j, u_k)(a_{23}(u, u_j)c_{211}(u, u_j) + a_{33}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{13}(u, u_k)c_{210}(u, u_j)a_{21}(u_j, u_k) \\
&+ a_{13}(u, u_j)(c_{29}(u, u_j)a_{13}(u_j, u_k) + c_{210}(u, u_j)a_{23}(u_j, u_k)) \\
&+ a_{23}(u, u_k)c_{212}(u, u_j)a_{21}(u_j, u_k) + a_{33}(u, u_k)c_{214}(u, u_j)a_{21}(u_j, u_k) \\
&+ a_{33}(u, u_j)c_{214}(u, u_j)a_{23}(u_j, u_k) + a_{23}(u, u_j)c_{212}(u, u_j)a_{23}(u_j, u_k) \\
&+ c_{210}(u, u_j)x_{14}(u, u_j)a_{25}(u_j, u_k) + c_{212}(u, u_j)x_{24}(u, u_j)a_{25}(u_j, u_k) \\
&+ c_{214}(u, u_j)x_{34}(u, u_j)a_{25}(u_j, u_k) + c_{23}(u, u_j)c_{212}(u_j, u_k) - Y_{212}(u, u_j)e_{05}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{21}(u_j, u_k)(a_{13}(u, u_k)c_{19}(u, u_j) + a_{23}(u, u_k)c_{111}(u, u_j)) \\
&+ c_{19}(u, u_j)(a_{13}(u, u_j)a_{23}(u_j, u_k) + x_{14}(u, u_j)a_{25}(u_j, u_k)) \\
&+ a_{13}(u_j, u_k)(a_{13}(u, u_j)c_{18}(u, u_j) + a_{23}(u, u_j)c_{110}(u, u_j)) \\
&+ a_{23}(u, u_j)c_{111}(u, u_j)a_{23}(u_j, u_k) + c_{111}(u, u_j)x_{24}(u, u_j)a_{25}(u_j, u_k) \\
&+ c_{13}(u, u_j)c_{212}(u_j, u_k) - Y_{110}(u, u_j)e_{05}(u_j, u_k)], \tag{B.38}
\end{aligned}$$

$$\begin{aligned}
Q_{11}^{\mathcal{W}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{11}(u_j, u_k)(a_{24}(u, u_k)c_{211}(u, u_j) + a_{34}(u, u_k)c_{213}(u, u_j)) \\
&+ a_{12}(u_j, u_k)(a_{24}(u, u_j)c_{211}(u, u_j) + a_{34}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{22}(u_j, u_k)(a_{24}(u, u_j)c_{212}(u, u_j) + a_{34}(u, u_j)c_{214}(u, u_j)) \\
&+ x_{11}(u_j, u)(a_{24}(u, u_k)c_{27}(u, u_j) + a_{34}(u, u_k)c_{28}(u, u_j)) \\
&+ a_{24}(u_j, u_k)(c_{212}(u, u_j)x_{26}(u, u_j) + c_{214}(u, u_j)x_{36}(u, u_j)) \\
&+ c_{24}(u, u_j)(c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) - Y_{214}(u, u_j)e_{04}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{24}(u, u_k)(c_{110}(u, u_j)a_{11}(u_j, u_k) + c_{17}(u, u_j)x_{11}(u_j, u)) \\
&+ a_{24}(u, u_j)c_{110}(u, u_j)a_{12}(u_j, u_k) \\
&+ c_{111}(u, u_j)(a_{24}(u, u_j)a_{22}(u_j, u_k) + x_{26}(u, u_j)a_{24}(u_j, u_k)) \\
&+ c_{14}(u, u_j)(c_{26}(u_j, u_k) + c_{29}(u_j, u_k)) - Y_{112}(u, u_j)e_{04}(u_j, u_k)], \quad (\text{B.39})
\end{aligned}$$

$$\begin{aligned}
Q_{12}^{\mathcal{W}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{11}(u_j, u_k)(a_{25}(u, u_k)c_{211}(u, u_j) + a_{35}(u, u_k)c_{213}(u, u_j)) \\
&+ a_{13}(u_j, u_k)(a_{24}(u, u_j)c_{211}(u, u_j) + a_{34}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{23}(u_j, u_k)(a_{24}(u, u_j)c_{212}(u, u_j) + a_{34}(u, u_j)c_{214}(u, u_j)) \\
&+ x_{11}(u_j, u)(a_{25}(u, u_k)c_{27}(u, u_j) + a_{35}(u, u_k)c_{28}(u, u_j)) \\
&+ a_{25}(u_j, u_k)(c_{212}(u, u_j)x_{26}(u, u_j) + c_{214}(u, u_j)x_{36}(u, u_j)) \\
&+ c_{24}(u, u_j)c_{210}(u_j, u_k) - Y_{214}(u, u_j)e_{05}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{25}(u, u_k)(c_{110}(u, u_j)a_{11}(u_j, u_k) + c_{17}(u, u_j)x_{11}(u_j, u)) \\
&+ a_{24}(u, u_j)c_{110}(u, u_j)a_{13}(u_j, u_k) \\
&+ c_{111}(u, u_j)(a_{24}(u, u_j)a_{23}(u_j, u_k) + x_{26}(u, u_j)a_{25}(u_j, u_k)) \\
&+ c_{14}(u, u_j)c_{210}(u_j, u_k) - Y_{112}(u, u_j)e_{05}(u_j, u_k)], \quad (\text{B.40})
\end{aligned}$$

$$\begin{aligned}
Q_{21}^{\mathcal{W}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{12}(u_j, u_k)(a_{25}(u, u_j)c_{211}(u, u_j) + a_{35}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{21}(u_j, u_k)(a_{24}(u, u_k)c_{212}(u, u_j) + a_{34}(u, u_k)c_{214}(u, u_j)) \\
&+ a_{22}(u_j, u_k)(a_{25}(u, u_j)c_{212}(u, u_j) + a_{35}(u, u_j)c_{214}(u, u_j)) \\
&+ a_{24}(u_j, u_k)(c_{212}(u, u_j)x_{27}(u, u_j) + c_{214}(u, u_j)x_{37}(u, u_j)) \\
&+ c_{24}(u, u_j)(c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) - Y_{215}(u, u_j)e_{04}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{25}(u, u_j)c_{110}(u, u_j)a_{12}(u_j, u_k) \\
&+ c_{111}(u, u_j)(a_{24}(u, u_k)a_{21}(u_j, u_k) + a_{25}(u, u_j)a_{22}(u_j, u_k) + x_{27}(u, u_j)a_{24}(u_j, u_k)) \\
&+ c_{14}(u, u_j)(c_{211}(u_j, u_k) + c_{27}(u_j, u_k)) - Y_{113}(u, u_j)e_{04}(u_j, u_k)], \quad (\text{B.41})
\end{aligned}$$

$$\begin{aligned}
Q_{22}^{\mathcal{W}}(u, u_j, u_k) &= k_{23}^+(u) \\
&\times [a_{13}(u_j, u_k)(a_{25}(u, u_j)c_{211}(u, u_j) + a_{35}(u, u_j)c_{213}(u, u_j)) \\
&+ a_{25}(u, u_k)c_{212}(u, u_j)a_{21}(u_j, u_k) + a_{35}(u, u_k)c_{214}(u, u_j)a_{21}(u_j, u_k) \\
&+ a_{23}(u_j, u_k)(a_{25}(u, u_j)c_{212}(u, u_j) + a_{35}(u, u_j)c_{214}(u, u_j)) \\
&+ c_{212}(u, u_j)x_{27}(u, u_j)a_{25}(u_j, u_k) + c_{214}(u, u_j)x_{37}(u, u_j)a_{25}(u_j, u_k) \\
&+ c_{24}(u, u_j)c_{212}(u_j, u_k) - Y_{215}(u, u_j)e_{05}(u_j, u_k)] \\
&+ k_{12}^+(u)[a_{25}(u, u_j)c_{110}(u, u_j)a_{13}(u_j, u_k) \\
&+ c_{111}(u, u_j)(a_{25}(u, u_k)a_{21}(u_j, u_k) + a_{25}(u, u_j)a_{23}(u_j, u_k) + x_{27}(u, u_j)a_{25}(u_j, u_k)) \\
&+ c_{14}(u, u_j)c_{212}(u_j, u_k) - Y_{113}(u, u_j)e_{05}(u_j, u_k)]. \tag{B.42}
\end{aligned}$$

We do not present the explicit expressions for the polynomials $Q_{jkl}^{\mathcal{X}, \mathcal{Y}, \mathcal{Z}}$ since they are very cumbersome and are not necessary in the determination of the g -coefficients. An additional systematic analysis of the commutation relations may allow us to write these coefficients in a more manageable form.